Problem 11

A home buyer wishes to borrow $250,000 at an interest rate of 6% to finance the purchase. Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow on a 20-year mortgage; on a 30-year mortgage.

(b) Determine the total interest paid during the term of the mortgage in each of the cases in part (a).

Solution

The amount of money $S(t)$ that the buyer has to pay changes in time due to two factors, the compound interest and his continuous payments. The rate of growth for compounding is $rS$, and the rate of decay due to the continuous payments is $k$.

$$\frac{dS}{dt} = rS - k$$

Bring $rS$ to the left side.

$$\frac{dS}{dt} - rS = -k$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp \left( \int (-r) \, ds \right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt} \frac{dS}{dt} - re^{-rt}S = -ke^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt}S) = -ke^{-rt}$$

Integrate both sides with respect to $t$.

$$e^{-rt}S = \frac{k}{r}e^{-rt} + C$$

Multiply both sides by $e^{rt}$.

$$S(t) = \frac{k}{r} + Ce^{rt}$$

Apply the initial condition $S(0) = 250,000$ to determine $C$.

$$S(0) = \frac{k}{r} + C = 250,000 \quad \rightarrow \quad C = 250,000 - \frac{k}{r}$$

Therefore, the amount of money the buyer has to pay after $t$ years is

$$S(t) = \frac{k}{r} + \left( 250,000 - \frac{k}{r} \right) e^{rt}$$

$$= \frac{k}{r} (1 - e^{rt}) + 250,000 e^{rt}.$$
Set \( r = 6\% = 0.06 \).

\[
S(t) = \frac{k}{0.06} (1 - e^{0.06t}) + 250000e^{0.06t}
\]

For a 20-year mortgage, \( S(t) = 0 \) at \( t = 20 \).

\[
0 = \frac{k}{0.06} (1 - e^{0.06 \cdot 20}) + 250000e^{0.06 \cdot 20} \quad \Rightarrow \quad k \approx \$21,465.19 \text{ dollars/year} \approx 1788.77 \text{ dollars/month}
\]

For a 30-year mortgage, \( S(t) = 0 \) at \( t = 30 \).

\[
0 = \frac{k}{0.06} (1 - e^{0.06 \cdot 30}) + 250000e^{0.06 \cdot 30} \quad \Rightarrow \quad k \approx \$17,970.50 \text{ dollars/year} \approx 1497.54 \text{ dollars/month}
\]

These values of \( k \) are how much the buyer has to pay per year (or month). For the 20-year mortgage, he pays a total of about \( $21,465.19 \times 20 \approx $429,303.83 \), so the total interest he pays is about

\[
$429,303.83 - $250,000.00 \approx $179,303.83.
\]

For the 30-year mortgage, he pays a total of \( $17,970.50 \times 30 \approx $539,115.13 \), so the total interest he pays is

\[
$539,115.13 - $250,000.00 \approx $289,115.13.
\]