Problem 12

A recent college graduate borrows \$150,000 at an interest rate of 6% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800 + 10t$, where t is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

Solution

Part (a)

The amount of money $S(t)$ that the graduate has to pay changes in time due to two factors, the compound interest and his continuous payments. The rate of growth for compounding is rS, and the rate of decay due to the continuous payments is $800 + 10t$.

$$\frac{dS}{dt} = rS - (800 + 10t)$$

It's important to note that the given interest rate $r = 6\%$ is annual (per year); we'll have to change this to a monthly rate so that the units in the equation are consistent. t is in units of months, and dS/dt is in units of dollars per month in this problem.

$$r = 6\,\frac{\%}{\text{year}} \times \frac{1\text{ year}}{12\text{ months}} = \frac{1}{2}\,\frac{\%}{\text{month}} = \frac{1}{200}\,\frac{1}{\text{month}}$$

Bring rS to the left side.

$$\frac{dS}{dt} - rS = -10t - 800$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t (-r)\,ds\right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt}\frac{dS}{dt} - re^{-rt}S = -10te^{-rt} - 800e^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt}S) = -10te^{-rt} - 800e^{-rt}$$

Integrate both sides with respect to t, using integration by parts on the right.

$$\begin{aligned}
e^{-rt}S &= \int^t (-10se^{-rs} - 800e^{-rs})\,ds \\
&= -10\int^t se^{-rs}\,ds - 800\int^t e^{-rs}\,ds \\
&= -10\int^t s\frac{d}{ds}\left(-\frac{1}{r}e^{-rs}\right)ds + \frac{800}{r}e^{-rt} + C \\
&= -10\left[s\left(-\frac{1}{r}e^{-rs}\right)\Big|^t - \int^t 1\cdot\left(-\frac{1}{r}e^{-rs}\right)ds\right] + \frac{800}{r}e^{-rt} + C \\
&= -10\left[t\left(-\frac{1}{r}e^{-rt}\right) - \frac{1}{r^2}e^{-rt}\right] + \frac{800}{r}e^{-rt} + C \\
&= \frac{10}{r^2}(rt+1)e^{-rt} + \frac{800}{r}e^{-rt} + C
\end{aligned}$$

Multiply both sides by e^{rt}.

$$S(t) = \frac{10}{r^2}(rt+1) + \frac{800}{r} + Ce^{rt} \tag{1}$$

Apply the initial condition $S(0) = 150\,000$ to determine C.

$$S(0) = \frac{10}{r^2} + \frac{800}{r} + C = 150\,000 \quad \rightarrow \quad \frac{10}{\frac{1}{200^2}} + \frac{800}{\frac{1}{200}} + C = 150\,000 \quad \rightarrow \quad C = -410\,000$$

Therefore, the amount of money the buyer has to pay after t months is

$$S(t) = \frac{10}{r^2}(rt+1) + \frac{800}{r} - 410\,000e^{rt}.$$

Set $r = 1/200 = 0.005$.

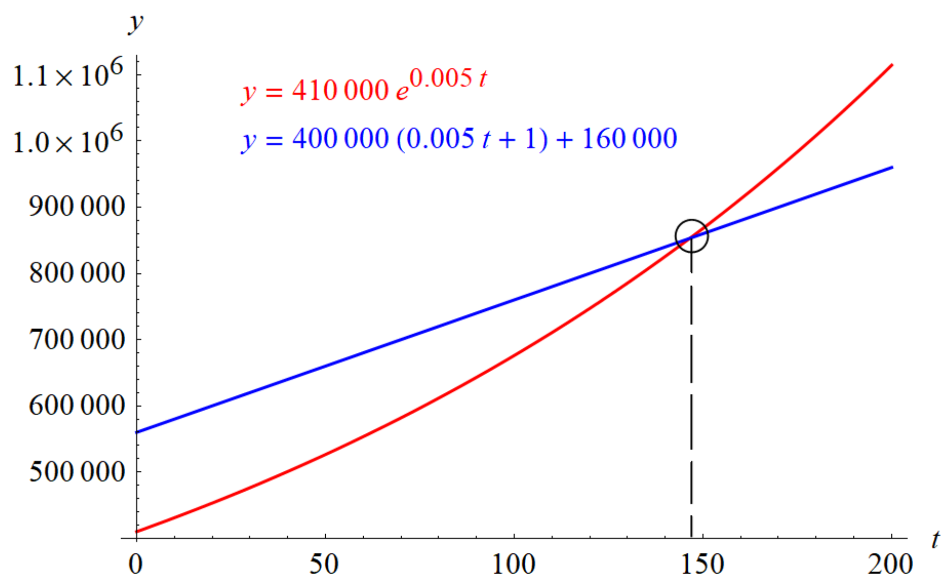
$$\begin{aligned}
S(t) &= \frac{10}{0.005^2}(0.005t+1) + \frac{800}{0.005} - 410\,000e^{0.005t} \\
&= 400\,000(0.005t+1) + 160\,000 - 410\,000e^{0.005t}
\end{aligned}$$

To find when the loan will be paid off, set $S(t) = 0$ and solve the resulting equation for t.

$$0 = 400\,000(0.005t+1) + 160\,000 - 410\,000e^{0.005t}$$

$$410\,000e^{0.005t} = 400\,000(0.005t+1) + 160\,000$$

Plot the functions on both sides on the same set of axes and find the value of t at which they intersect.



We see that $t \approx 147$ months.

Part (b)

Return to equation (1).

$$S(t) = \frac{10}{r^2}(rt+1) + \frac{800}{r} + Ce^{rt}$$

Use the initial condition $S(0) = S_0$, where S_0 is the initial loan size. Our aim in this part is to determine the largest value it can take.

$$S(0) = \frac{10}{r^2} + \frac{800}{r} + C = S_0 \quad \rightarrow \quad \frac{10}{\frac{1}{200^2}} + \frac{800}{\frac{1}{200}} + C = S_0 \quad \rightarrow \quad C = S_0 - 560\,000$$

As a result,

$$S(t) = \frac{10}{r^2}(rt+1) + \frac{800}{r} + (S_0 - 560\,000)e^{rt}.$$

Set $r = 1/200 = 0.005$.

$$\begin{aligned} S(t) &= \frac{10}{0.005^2}(0.005t+1) + \frac{800}{0.005} + (S_0 - 560\,000)e^{0.005t} \\ &= 400\,000(0.005t+1) + 160\,000 + (S_0 - 560\,000)e^{0.005t} \end{aligned}$$

Set $S(t) = 0$ and $t = 20 \times 12 = 240$ to find the maximum loan S_0 that can be paid in 20 years.

$$0 = 400\,000(0.005 \cdot 240 + 1) + 160\,000 + (S_0 - 560\,000)e^{0.005 \cdot 240}$$

Therefore,

$$S_0 \approx \$246\,758.02.$$