Problem 13

An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby.[3] This is a means of determining the age of certain wood and plant remains, and hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years[4]), measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14.

(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

Solution

The rate that the mass of carbon-14 decreases is assumed to be proportional to how much is present at any given time.

$$\frac{dQ}{dt} \propto -Q$$

Change this proportionality to an equation by introducing a constant r on the right side.

$$Q' = -rQ$$

Divide both sides by Q.

$$\frac{Q'}{Q} = -r$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt} \ln Q = -r$$

Integrate both sides with respect to t.

$$\ln Q = -rt + C$$

[3] Willard F. Libby (1908–1980) was born in rural Colorado and received his education at the University of California at Berkeley. He developed the method of radiocarbon dating beginning in 1947 while he was at the University of Chicago. For this work he was awarded the Nobel Prize in chemistry in 1960.

[4] *McGraw-Hill Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

Exponentiate both sides.

$$\begin{aligned} Q(t) &= e^{-rt+C} \\ &= e^C e^{-rt} \end{aligned}$$

Use a new constant A for e^C.

$$Q(t) = Ae^{-rt}$$

Suppose that there is a certain amount of mass Q_0 initially. Then the initial condition is $Q(0) = Q_0$. Use it to determine A.

$$Q(0) = A = Q_0$$

Consequently, the mass decays exponentially.

$$Q(t) = Q_0 e^{-rt}$$

r can be determined with knowledge of the half-life. For carbon-14 specifically, we know that half the initial mass is lost after 5730 years.

$$\frac{Q_0}{2} = Q_0 e^{-r(5730)}$$

Solve this equation for r.

$$e^{-5730r} = \frac{1}{2}$$

$$\ln e^{-5730r} = \ln \frac{1}{2}$$

$$-5730r = \ln \frac{1}{2}$$

$$r = -\frac{1}{5730} \ln \frac{1}{2} = \frac{\ln 2}{5730} \approx 0.0001210 \ \frac{1}{\text{year}}$$

Therefore, the mass of a sample of carbon-14 is

$$Q(t) = Q_0 \exp\left(-\frac{\ln 2}{5730} t\right),$$

where t is in years. If some remains have 20% of the original amount of carbon-14, then $Q(t) = 0.2Q_0$. Solve the resulting equation for t to determine how old the remains are.

$$0.2Q_0 = Q_0 \exp\left(-\frac{\ln 2}{5730} t\right)$$

$$\exp\left(-\frac{\ln 2}{5730} t\right) = 0.2$$

$$\ln \exp\left(-\frac{\ln 2}{5730} t\right) = \ln 0.2$$

$$-\frac{\ln 2}{5730} t = \ln \frac{1}{5}$$

$$t = \frac{\ln 5}{\ln 2} 5730 \approx 13\,305 \text{ years}$$