Problem 14

Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

\[ \frac{dy}{dt} = \frac{(0.5 + \sin t)}{5} y. \]

(a) If \( y(0) = 1 \), find (or estimate) the time \( \tau \) at which the population has doubled. Choose other initial conditions and determine whether the doubling time \( \tau \) depends on the initial population.

(b) Suppose that the growth rate is replaced by its average value 1/10. Determine the doubling time \( \tau \) in this case.

(c) Suppose that the term \( \sin t \) in the differential equation is replaced by \( \sin 2\pi t \); that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time \( \tau \)?

(d) Plot the solutions obtained in parts (a), (b), and (c) on a single set of axes.

Solution

Part (a)

The ODE describing the population is separable.

\[ \frac{dy}{dt} = \frac{0.5 + \sin t}{5} y \]

Separate variables.

\[ \frac{dy}{y} = \frac{0.5 + \sin t}{5} dt \]

Integrate both sides.

\[ \ln y = 0.1t - 0.2 \cos t + C \]

Exponentiate both sides.

\[ y(t) = e^{0.1t - 0.2 \cos t + C} \]

\[ y(t) = e^C e^{0.1t - 0.2 \cos t} \]

Use a new constant \( A \) for \( e^C \).

\[ y(t) = Ae^{0.1t - 0.2 \cos t} \]

Apply the initial condition \( y(0) = 1 \) to determine \( A \).

\[ y(0) = Ae^{-0.2} = 1 \quad \rightarrow \quad A = e^{0.2} \]

Therefore, the population is

\[ y(t) = e^{0.2} e^{0.1t - 0.2 \cos t} \]

\[ y(t) = e^{0.1t + 0.2(1 - \cos t)} \]

Plot \( y(t) \) versus \( t \) and find when the initial population doubles to \( y = 2 \).
From the graph we see that the population doubles at about $\tau \approx 3$. Rather than $y(0) = 1$, let’s apply $y(0) = 3$ to the general solution in equation (1).

$$y(0) = Ae^{-0.2t} = 3 \quad \rightarrow \quad A = 3e^{0.2t}$$

In this case the population is

$$y(t) = 3e^{0.2t}e^{0.1t-0.2\cos t} = 3e^{0.1t+0.2(1-\cos t)}.$$ 

The population still doubles at $\tau \approx 3$. 

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Part (b)

The average of the function of $t$ in front of $y$ in the ODE over one period is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{0.5 + \sin t}{5} \, dt = \frac{1}{10}.\]

Use this value in the ODE instead.

$$\frac{dy}{dt} = \frac{1}{10}y$$

Divide both sides by $y$.

$$\frac{dy}{dt} \frac{1}{y} = \frac{1}{10}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt} \ln y = \frac{1}{10}$$

Integrate both sides with respect to $t$.

$$\ln y = \frac{1}{10}t + C_1$$

Exponentiate both sides.

$$y(t) = e^{t/10 + C_1} = e^{C_1} e^{t/10}$$

Use a new constant $A_1$ for $e^{C_1}$.

$$y(t) = A_1 e^{t/10}$$

Use the initial condition $y(0) = y_0$ to determine $A_1$. The point of using a general initial condition $y(0) = y_0$ is to show that the doubling constant is independent of $y_0$, the initial population.

$$y(0) = A_1 = y_0$$

So then

$$y(t) = y_0 e^{t/10}.$$ 

Set $y(t) = 2y_0$ and solve for $t = \tau$ to determine the doubling constant.

$$2y_0 = y_0 e^{\tau/10}$$

$$2 = e^{\tau/10}$$

$$\ln 2 = \ln e^{\tau/10}$$

$$\ln 2 = \frac{\tau}{10}$$

$$\tau = 10 \ln 2 \approx 6.93$$

For $y_0 = 1$, the population is

$$y(t) = e^{t/10}.$$
**Part (c)**

The ODE describing the population is still separable.

\[
\frac{dy}{dt} = \frac{0.5 + \sin 2\pi t}{5} y
\]

Separate variables.

\[
\frac{dy}{y} = \frac{0.5 + \sin 2\pi t}{5} dt
\]

Integrate both sides.

\[
\ln y = \frac{1}{10} t - \frac{1}{10\pi} \cos 2\pi t + C_2
\]

Exponentiate both sides.

\[
y(t) = \exp \left( \frac{1}{10} t - \frac{1}{10\pi} \cos 2\pi t + C_2 \right)
= e^{C_2} \exp \left( \frac{1}{10} t - \frac{1}{10\pi} \cos 2\pi t \right)
\]

Use a new constant \( A_2 \) for \( e^{C_2} \).

\[
y(t) = A_2 \exp \left( \frac{1}{10} t - \frac{1}{10\pi} \cos 2\pi t \right)
\]

Apply the initial condition \( y(0) = 1 \) to determine \( A_2 \).

\[
y(0) = A_2 \exp \left( -\frac{1}{10\pi} \right) = 1 \quad \rightarrow \quad A_2 = \exp \left( \frac{1}{10\pi} \right)
\]

Therefore, the population is

\[
y(t) = \exp \left( \frac{1}{10\pi} \right) \exp \left( \frac{1}{10} t - \frac{1}{10\pi} \cos 2\pi t \right)
= \exp \left[ \frac{1}{10} t + \frac{1}{10\pi} (1 - \cos 2\pi t) \right].
\]
Plot $y(t)$ versus $t$ to find when the initial population doubles to $y = 2$.

From the graph we see that the population doubles at about $\tau \approx 6.4$.

**Part (d)**

$y(t) = e^{0.1t+0.2(1-\cos t)}$

$y(t) = e^{t/10}$

$y(t) = \exp\left[\frac{1}{10} t + \frac{1}{10\pi} (1-\cos 2\pi t)\right]$