Problem 17

Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann[5] law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \tag{i}$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T, then solutions of Eq. (i) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \tag{ii}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2.0 \times 10^{-12}$ K^{-3}/s.

(a) Determine the temperature of the body at any time by solving Eq. (ii).

(b) Plot the graph of u versus t.

(c) Find the time τ at which $u(\tau) = 600$—that is, twice the ambient temperature. Up to this time the error in using Eq. (ii) to approximate the solutions of Eq. (i) is no more than 1%.

Solution

Part (a)

Solve Eq. (ii) by separating variables.

$$\frac{du}{u^4} = -\alpha\, dt$$

Integrate both sides.

$$-\frac{1}{3u^3} = -\alpha t + C$$

Solve for u.

$$3u^3 = \frac{1}{\alpha t - C}$$

So then

$$u(t) = \frac{1}{\sqrt[3]{3\alpha t - 3C}}.$$

Apply the initial condition $u(0) = 2000$ to determine the constant.

$$u(0) = \frac{1}{\sqrt[3]{-3C}} = 2000 \quad \rightarrow \quad -3C = \frac{1}{2000^3} = 1.25 \times 10^{-10}$$

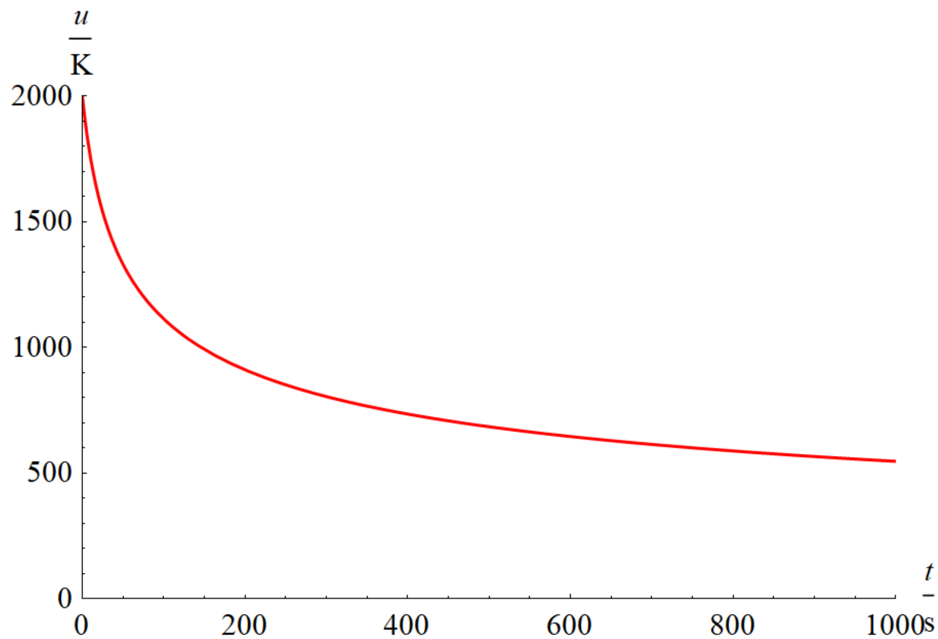
Therefore, plugging in $\alpha = 2.0 \times 10^{-12}$,

$$u(t) = \frac{1}{\sqrt[3]{3(2.0 \times 10^{-12})t + 1.25 \times 10^{-10}}} = \frac{10\,000}{\sqrt[3]{6t + 125}} = \frac{2000}{\sqrt[3]{0.048t + 1}}.$$

[5] Jozef Stefan (1835–1893), professor of physics at Vienna, stated the radiation law on empirical grounds in 1879. His student Ludwig Boltzmann (1844–1906) derived it theoretically from the principles of thermodynamics in 1884. Boltzmann is best known for his pioneering work in statistical mechanics.

Part (b)

This plot illustrates the temperature of an object that is losing heat by radiation.



Part (c)

Set $u(t) = 600$ and solve for $t = \tau$ to find the time that the temperature of the object is double the ambient temperature.

$$600 = \frac{2000}{\sqrt[3]{0.048\tau + 1}}$$

$$\frac{3}{10} = \frac{1}{\sqrt[3]{0.048\tau + 1}}$$

$$\sqrt[3]{0.048\tau + 1} = \frac{10}{3}$$

$$0.048\tau + 1 = \frac{1000}{27}$$

$$\tau = \frac{973}{27 \cdot 0.048} \approx 751 \text{ seconds}$$

This is about 12 minutes and 31 seconds.