Problem 18

Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton’s law of cooling, $u$ satisfies the differential equation

$$\frac{du}{dt} = -k[u - T(t)], \tag{i}$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos \omega t$.

(a) Solve Eq. (i) and express $u(t)$ in terms of $t, k, T_0, T_1, \text{ and } \omega$. Observe that part of your solution approaches zero as $t$ becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

(b) Suppose that $t$ is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60^\circ \text{F}, T_1 = 15^\circ \text{F}, \text{ and } k = 0.2/\text{h}$. Draw graphs of $S(t)$ and $T(t)$ versus $t$ on the same axes. From your graph estimate the amplitude $R$ of the oscillatory part of $S(t)$. Also estimate the time lag $\tau$ between corresponding maxima of $T(t)$ and $S(t)$.

(c) Let $k, T_0, T_1, \text{ and } \omega$ now be unspecified. Write the oscillatory part of $S(t)$ in the form $R \cos[\omega(t - \tau)]$. Use trigonometric identities to find expressions for $R$ and $\tau$. Let $T_1$ and $\omega$ have the values given in part (b), and plot graphs of $R$ and $\tau$ versus $k$.

Solution

Part (a)

Distribute $k$ in the ODE.

$$\frac{du}{dt} = -ku + kT(t)$$

Bring $ku$ to the left side and substitute $T(t) = T_0 + T_1 \cos \omega t$.

$$\frac{du}{dt} + ku = kT_0 + kT_1 \cos \omega t$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp \left( \int k \, ds \right) = e^{kt}$$

Proceed with the multiplication.

$$e^{kt} \frac{du}{dt} + ke^{kt} u = kT_0 e^{kt} + kT_1 e^{kt} \cos \omega t$$

The left side can be written as $d/dt(\text{I}u)$ by the product rule.

$$\frac{d}{dt}(e^{kt} u) = kT_0 e^{kt} + kT_1 e^{kt} \cos \omega t$$

Integrate both sides with respect to $t$.

$$e^{kt} u = \int (kT_0 e^{ks} + kT_1 e^{ks} \cos \omega s) \, ds + C$$

www.stemjock.com
Distribute the integral and bring the constants in front of them.

\[ e^{kt}u = T_0e^{kt} + kT_1 \int^t e^{ks} \cos \omega s \, ds + C \]  

(1)

Use integration by parts twice for the remaining integral.

\[
\int^t e^{ks} \cos \omega s \, ds = \int^t e^{ks} \frac{d}{ds} \left( \frac{1}{\omega} \sin \omega s \right) \, ds
\]

\[
= e^{ks} \left( \frac{1}{\omega} \sin \omega s \right) \bigg|^t_0 - \int^t ke^{ks} \frac{1}{\omega} \sin \omega s \, ds
\]

\[
= \frac{e^{kt}}{\omega} \sin \omega t - \frac{k}{\omega} \int^t e^{ks} \frac{d}{ds} \left( \frac{1}{\omega} \cos \omega s \right) \, ds
\]

\[
= \frac{e^{kt}}{\omega} \sin \omega t - \frac{k}{\omega} \left[ e^{ks} \left( \frac{1}{\omega} \cos \omega s \right) \right]^t_0 - \int^t ke^{ks} \left( -\frac{1}{\omega} \cos \omega s \right) \, ds
\]

\[
= \frac{e^{kt}}{\omega} \sin \omega t - \frac{k}{\omega} e^{kt} \cos \omega t + \frac{k}{\omega} \int^t e^{ks} \cos \omega s \, ds
\]

Solve for the integral.

\[
\left( 1 + \frac{k^2}{\omega^2} \right) \int^t e^{ks} \cos \omega s \, ds = \frac{e^{kt}}{\omega^2} (k \cos \omega t + \omega \sin \omega t)
\]

As a result,

\[
\int^t e^{ks} \cos \omega s \, ds = \frac{e^{kt}}{k^2 + \omega^2} (k \cos \omega t + \omega \sin \omega t).
\]

Substitute this result into equation (1).

\[
e^{kt}u = T_0e^{kt} + kT_1 \frac{e^{kt}}{k^2 + \omega^2} (k \cos \omega t + \omega \sin \omega t) + C
\]

Divide both sides by \( e^{kt} \) to get the general solution.

\[
u(t) = T_0 + T_1 \underbrace{\frac{k}{k^2 + \omega^2} (k \cos \omega t + \omega \sin \omega t)}_{\text{steady}} + \underbrace{\frac{Ce^{-kt}}{k^2 + \omega^2}}_{\text{transient}}
\]

It can be thought of as the sum of a steady term, one that remains as \( t \to \infty \), and a transient term, one that dies out as \( t \to \infty \). Define \( S(t) \) to be the steady part.

\[
S(t) = T_0 + T_1 \frac{k}{k^2 + \omega^2} (k \cos \omega t + \omega \sin \omega t)
\]
Part (b)

Let \( \omega = \pi/12 \), \( T_0 = 60 \), \( T_1 = 15 \), and \( k = 0.2 \). Then

\[
S(t) = 60 + 15 \left( 0.2 \cos \frac{\pi t}{12} + \frac{\pi}{12} \sin \frac{\pi t}{12} \right)
\]

\[
= 60 + \frac{3}{5 \left( \frac{1}{25} + \frac{\pi^2}{144} \right)} \cos \frac{\pi t}{12} + \frac{\pi}{4 \left( \frac{1}{25} + \frac{\pi^2}{144} \right)} \sin \frac{\pi t}{12}
\]

and

\[
T(t) = 60 + 15 \cos \frac{\pi t}{12}
\]

Below is a plot of \( S(t) \) and \( T(t) \) versus \( t \).

There’s no need to estimate the amplitude from the graph. Let

\[
A \cos \phi = \frac{3}{5 \left( \frac{1}{25} + \frac{\pi^2}{144} \right)}
\]

\[
A \sin \phi = \frac{\pi}{4 \left( \frac{1}{25} + \frac{\pi^2}{144} \right)}
\]

and solve the two equations for the two unknowns, \( A \) and \( \phi \). Divide both sides of the second equation by those of the first.

\[
\tan \phi = \frac{5\pi}{12}
\]
Draw the implied triangle to determine $\cos \phi$.

Now $A$ can be found.

$$A \left( \frac{12}{\sqrt{25\pi^2 + 144}} \right) = \frac{3}{5 \left( \frac{1}{25} + \frac{\pi^2}{144} \right)} \quad \rightarrow \quad A = \frac{180}{\sqrt{25\pi^2 + 144}}$$

Consequently,

$$S(t) = 60 + A \cos \phi \cos \frac{\pi t}{12} + A \sin \phi \sin \frac{\pi t}{12}$$

$$= 60 + \frac{180}{\sqrt{25\pi^2 + 144}} \cos \left( \frac{\pi t}{12} - \tan^{-1} \frac{5\pi}{12} \right)$$

$$= 60 + \frac{180}{\sqrt{25\pi^2 + 144}} \cos \left[ \frac{\pi}{12} \left( t - \frac{12}{\pi} \tan^{-1} \frac{5\pi}{12} \right) \right].$$

The amplitude is

$$R = \frac{180}{\sqrt{25\pi^2 + 144}} \approx 9.11^\circ \text{F},$$

and the graph of $T(t)$ lags behind that of $S(t)$ by

$$\tau = \frac{12}{\pi} \tan^{-1} \frac{5\pi}{12} \approx 3.51 \text{ hours.}$$
Part (c)

Recall that \( S(t) \) is the steady part of the temperature.

\[
S(t) = T_0 + T_1 \frac{k}{k^2 + \omega^2} (k \cos \omega t + \omega \sin \omega t)
\]

\[
= T_0 + T_1 \frac{k^2}{k^2 + \omega^2} \cos \omega t + T_1 \frac{k\omega}{k^2 + \omega^2} \sin \omega t
\]

Let

\[
R \cos \psi = T_1 \frac{k^2}{k^2 + \omega^2}
\]

\[
R \sin \psi = T_1 \frac{k\omega}{k^2 + \omega^2}
\]

and solve the two equations for the two unknowns, \( R \) and \( \psi \). Divide both sides of the second equation by those of the first.

\[
\tan \psi = \frac{\omega}{k}
\]

Draw the implied triangle to determine \( \cos \psi \).

Now \( R \) can be found.

\[
R \left( \frac{k}{\sqrt{k^2 + \omega^2}} \right) = T_1 \frac{k^2}{k^2 + \omega^2} \quad \rightarrow \quad R = T_1 \frac{k}{\sqrt{k^2 + \omega^2}}
\]

Therefore,

\[
S(t) = T_0 + R \cos \psi \cos \omega t + R \sin \psi \sin \omega t
\]

\[
= T_0 + R \cos(\omega t - \psi)
\]

\[
= T_0 + T_1 \frac{k}{\sqrt{k^2 + \omega^2}} \cos \left( \omega t - \tan^{-1} \frac{\omega}{k} \right)
\]

\[
= T_0 + T_1 \frac{k}{\sqrt{k^2 + \omega^2}} \cos \left( \omega \left( t - \frac{1}{\omega} \tan^{-1} \frac{\omega}{k} \right) \right)
\]

and

\[
\tau = \frac{1}{\omega} \tan^{-1} \frac{\omega}{k}.
\]
If $T_1 = 15$ and $\omega = \pi/12$, then

$$R = 15 \frac{k}{\sqrt{k^2 + \frac{\pi^2}{144}}}$$

$$\tau = \frac{12}{\pi} \tan^{-1} \frac{\pi}{12k}.$$