Problem 24

A rocket sled having an initial speed of 150 mi/h is slowed by a channel of water. Assume that during the braking process, the acceleration $a$ is given by $a(v) = -\mu v^2$, where $v$ is the velocity and $\mu$ is a constant.

(a) As in Example 4 in the text, use the relation $\frac{dv}{dt} = v(\frac{dv}{dx})$ to write the equation of motion in terms of $v$ and $x$.

(b) If it requires a distance of 2000 ft to slow the sled to 15 mi/h, determine the value of $\mu$.

(c) Find the time $\tau$ required to slow the sled to 15 mi/h.

Solution

Part (a)

The acceleration is given by

$$a = -\mu v^2.$$ 

Replace $a$ with $\frac{dv}{dt}$.

$$\frac{dv}{dt} = -\mu v^2$$

Use the chain rule to write $v$ in terms of $x$:

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx}v.$$ 

As a result, the previous equation becomes

$$\frac{dv}{dx}v = -\mu v^2.$$ 

Divide both sides by $v^2$

$$\frac{dv}{dx} = -\mu$$

and then rewrite the left side as a derivative of a logarithm.

$$\frac{d}{dx} \ln |v| = -\mu$$

If we take the positive $x$-axis to point in the direction the sled is moving, then the absolute value sign can be dropped because $v$ will be positive. Integrate both sides with respect to $x$.

$$\ln v = -\mu x + C$$

Exponentiate both sides.

$$v(x) = e^{-\mu x + C} = e^C e^{-\mu x}$$

Use a new constant $A$ for $e^C$.

$$v(x) = Ae^{-\mu x}$$

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Use the fact that the initial speed is 150 mi/h to determine $A$.

$$v(0) = A = 150$$

So then

$$v(x) = 150e^{-\mu x}.$$  

**Part (b)**

Now use the fact that the sled requires a distance of 2000 ft to slow to 15 mi/h. Set $v(x) = 15$ and $x = 2000$ and solve for $\mu$.

$$15 = 150e^{-2000\mu}$$

$$\frac{1}{10} = e^{-2000\mu}$$

$$\ln \frac{1}{10} = -2000\mu$$

$$\mu = \frac{\ln 10}{2000} \times \frac{5280 \text{ ft}}{1 \text{ mi}} = \frac{66}{25} \ln 10 \approx 6.08 \frac{1}{\text{mi}}$$

Therefore,

$$v(x) = 150 \exp \left( -\frac{66 \ln 10}{25} x \right),$$

where $x$ is in miles and $v$ is in miles per hour.

**Part (c)**

Replace $v(x)$ with $dx/dt$ and solve the resulting ODE by separating variables.

$$\frac{dx}{dt} = 150 \exp \left( -\frac{66 \ln 10}{25} x \right)$$

$$\exp \left( \frac{66 \ln 10}{25} x \right) dx = 150 dt$$

Integrate both sides.

$$\frac{25}{66 \ln 10} \exp \left( \frac{66 \ln 10}{25} x \right) = 150t + C_2$$

Apply the initial condition $x(0) = 0$ to determine $C_2$.

$$\frac{25}{66 \ln 10} = C_2$$

The previous equation is then

$$\frac{25}{66 \ln 10} \exp \left( \frac{66 \ln 10}{25} x \right) = 150t + \frac{25}{66 \ln 10}.$$  

Solve for $t$.

$$150t = \frac{25}{66 \ln 10} \left[ \exp \left( \frac{66 \ln 10}{25} x \right) - 1 \right]$$

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Divide both sides by 150.

\[ t = \frac{1}{396 \ln 10} \left[ \exp \left( \frac{66 \ln 10}{25} x \right) - 1 \right]. \]

Therefore, the time \( \tau \) it takes for the sled to go

\[ x = 2000 \text{ ft} \times \frac{1 \text{ mi}}{5280 \text{ ft}} = \frac{25}{66} \text{ mi} \]

is

\[ \tau = \frac{1}{44 \ln 10} \text{ h} \times \frac{3600 \text{ seconds}}{1 \text{ hour}} \approx 35.53 \text{ seconds}. \]