Problem 25

A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.

(a) Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.

(b) Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right],$$
$$x_m = \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right].$$

(c) Show that the quantity kv_0/mg is dimensionless.

Solution

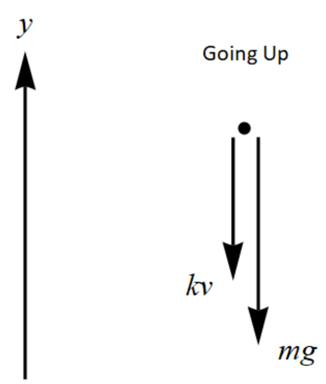
According to Newton's second law, the equation of motion for the mass is

$$\sum \mathbf{F} = m\mathbf{a}.$$

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the y-direction is relevant.

$$\sum F_y = ma_y$$

Draw the free-body diagram for the mass as it's travelling upward.



Now the equation of motion can be written.

$$-kv - mg = ma_y$$

Replace a_y with dv/dt, bring kv to the other side, and divide both sides by m.

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{k}{m}\,ds\right) = e^{kt/m}$$

Proceed with the multiplication.

$$e^{kt/m}\frac{dv}{dt} + \frac{k}{m}e^{kt/m}v = -ge^{kt/m}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{kt/m}v) = -ge^{kt/m}$$

Integrate both sides with respect to t.

$$e^{kt/m}v = -\frac{mg}{k}e^{kt/m} + C_1$$

Divide both sides by $e^{kt/m}$.

$$v(t) = -\frac{mg}{k} + C_1e^{-kt/m}$$

Apply the initial condition $v(0) = v_0$ to determine C_1.

$$v(0) = -\frac{mg}{k} + C_1 = v_0 \quad \rightarrow \quad C_1 = v_0 + \frac{mg}{k}$$

Therefore,

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Now replace $v(t)$ with dy/dt.

$$\frac{dy}{dt} = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}$$

Integrate both sides with respect to t.

$$y(t) = -\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-kt/m} + C_2$$

Use the initial condition $y(0) = 0$ to determine C_2.

$$y(0) = -\frac{m}{k}\left(v_0 + \frac{mg}{k}\right) + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)$$

Therefore,

$$\begin{aligned} y(t) &= -\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-kt/m} + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) \\ &= -\frac{mg}{k}t + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)(1 - e^{-kt/m}). \end{aligned}$$

Set $v(t) = 0$ and solve for the time $t = t_m$ at which the ball reaches its maximum and comes to a stop.

$$0 = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt_m/m}$$

$$\begin{aligned} e^{-kt_m/m} &= \frac{\frac{mg}{k}}{v_0 + \frac{mg}{k}} \\ \ln e^{-kt_m/m} &= \ln \frac{\frac{mg}{k}}{v_0 + \frac{mg}{k}} \\ -\frac{kt_m}{m} &= \ln \frac{1}{\frac{kv_0}{mg} + 1} \\ t_m &= \frac{m}{k} \ln\left(1 + \frac{kv_0}{mg}\right) \end{aligned}$$

The Taylor series expansion of $\ln(1+x)$ about $x = 0$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots,$$

so as long as kv_0/mg is really small ($kv_0/mg \ll 1$), t_m can be approximated well by

$$\begin{aligned} t_m &\approx \frac{m}{k}\left[\frac{kv_0}{mg} - \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^3 - \frac{1}{4}\left(\frac{kv_0}{mg}\right)^4 + \cdots\right] \\ &\approx \frac{v_0}{g}\left[1 - \frac{1}{2}\left(\frac{kv_0}{mg}\right) + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \frac{1}{4}\left(\frac{kv_0}{mg}\right)^3 + \cdots\right]. \end{aligned}$$

To find the maximum height, plug t_m into the formula for $y(t)$.

$$\begin{aligned} y(t_m) &= -\frac{mg}{k}t_m + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)(1 - e^{-kt_m/m}) \\ &= -\frac{mg}{k}\frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left\{1 - \exp\left[-\frac{k}{m}\frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right)\right]\right\} \\ &= -\frac{m^2g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left[1 - \exp\left(-\ln\frac{mg + kv_0}{mg}\right)\right] \\ &= -\frac{m^2g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - \frac{mg}{mg + kv_0}\right) \\ &= -\frac{m^2g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{m}{k}\frac{kv_0 + mg}{k}\frac{kv_0}{mg + kv_0} \\ &= -\frac{m^2g}{k^2}\ln\left(1 + \frac{kv_0}{mg}\right) + \frac{mv_0}{k} \end{aligned}$$

As long as kv_0/mg is really small, that is, $kv_0/mg \ll 1$, we can use the Taylor series expansion for $\ln(1 + kv_0/mg)$ to approximate the height.

$$\begin{aligned} y(t_m) &\approx -\frac{m^2g}{k^2}\left[\frac{kv_0}{mg} - \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^3 - \frac{1}{4}\left(\frac{kv_0}{mg}\right)^4 + \cdots\right] + \frac{mv_0}{k} \\ &\approx \underbrace{-\frac{m^2g}{k^2}\frac{kv_0}{mg} + \frac{mv_0}{k}}_{=0} + \frac{m^2g}{k^2}\left[\frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \frac{1}{3}\left(\frac{kv_0}{mg}\right)^3 + \frac{1}{4}\left(\frac{kv_0}{mg}\right)^4 - \cdots\right] \\ &\approx \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\left(\frac{kv_0}{mg}\right) + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right] \end{aligned}$$

Part (c)

Because kv is a force, the product of k and v must be in Newtons.

$$\begin{aligned}[kv] &= \text{N} \\ [k]\frac{\text{m}}{\text{s}} &= \text{kg}\cdot\frac{\text{m}}{\text{s}^2} \\ [k] &= \frac{\text{kg}}{\text{s}}\end{aligned}$$

Now we can look at the units of the ratio.

$$\left[\frac{kv_0}{mg}\right] = \frac{\frac{\text{kg}}{\text{s}}\cdot\frac{\text{m}}{\text{s}}}{\text{kg}\cdot\frac{\text{m}}{\text{s}^2}} = \frac{\text{kg}\cdot\frac{\text{m}}{\text{s}^2}}{\text{kg}\cdot\frac{\text{m}}{\text{s}^2}} = 1$$

Therefore, kv_0/mg is dimensionless.