Problem 25

A body of constant mass $m$ is projected vertically upward with an initial velocity $v_0$ in a medium offering a resistance $k|v|$, where $k$ is a constant. Neglect changes in the gravitational force.

(a) Find the maximum height $x_m$ attained by the body and the time $t_m$ at which this maximum height is reached.

(b) Show that if $kv_0/mg < 1$, then $t_m$ and $x_m$ can be expressed as

$$t_m = \frac{v_0}{g} \left[ 1 - \frac{1}{2} \frac{kv_0}{mg} + \frac{1}{3} \left( \frac{kv_0}{mg} \right)^2 - \cdots \right],$$

$$x_m = \frac{v_0^2}{2g} \left[ 1 - \frac{2}{3} \frac{kv_0}{mg} + \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 - \cdots \right].$$

(c) Show that the quantity $kv_0/mg$ is dimensionless.

Solution

According to Newton’s second law, the equation of motion for the mass is

$$\sum \mathbf{F} = m \mathbf{a}.$$  

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the $y$-direction is relevant.

$$\sum F_y = ma_y$$

Draw the free-body diagram for the mass as it’s travelling upward.

Now the equation of motion can be written.

$$-kv - mg = ma_y$$

Replace $a_y$ with $dv/dt$, bring $kv$ to the other side, and divide both sides by $m$.

$$\frac{dv}{dt} + \frac{k}{m} v = -g$$
This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$ I = \exp \left( \int \frac{k}{m} \, ds \right) = e^{kt/m} $$

Proceed with the multiplication.

$$ e^{kt/m} \frac{dv}{dt} + \frac{k}{m} e^{kt/m} v = -ge^{kt/m} $$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$ \frac{d}{dt}(e^{kt/m} v) = -ge^{kt/m} $$

Integrate both sides with respect to $t$.

$$ e^{kt/m} v = -\frac{mg}{k} e^{kt/m} + C_1 $$

Divide both sides by $e^{kt/m}$.

$$ v(t) = -\frac{mg}{k} + C_1 e^{-kt/m} $$

Apply the initial condition $v(0) = v_0$ to determine $C_1$.

$$ v(0) = -\frac{mg}{k} + C_1 = v_0 \rightarrow C_1 = v_0 + \frac{mg}{k} $$

Therefore,

$$ v(t) = -\frac{mg}{k} + \left( v_0 + \frac{mg}{k} \right) e^{-kt/m}. $$

Now replace $v(t)$ with $dy/dt$.

$$ \frac{dy}{dt} = -\frac{mg}{k} + \left( v_0 + \frac{mg}{k} \right) e^{-kt/m} $$

Integrate both sides with respect to $t$.

$$ y(t) = -\frac{mg}{k} t - \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) e^{-kt/m} + C_2 $$

Use the initial condition $y(0) = 0$ to determine $C_2$.

$$ y(0) = -\frac{m}{k} \left( v_0 + \frac{mg}{k} \right) + C_2 = 0 \rightarrow C_2 = \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) $$

Therefore,

$$ y(t) = -\frac{mg}{k} t - \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) e^{-kt/m} + \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) $$

$$ = -\frac{mg}{k} t + \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) (1 - e^{-kt/m}). $$

Set $v(t) = 0$ and solve for the time $t = t_m$ at which the ball reaches its maximum and comes to a stop.

$$ 0 = -\frac{mg}{k} + \left( v_0 + \frac{mg}{k} \right) e^{-kt_m/m} $$
\[ e^{-kt_m/m} = \frac{mg}{k} \frac{k}{v_0 + \frac{mg}{k}} \]
\[
\ln e^{-kt_m/m} = \ln \frac{mg}{k} \frac{k}{v_0 + \frac{mg}{k}} \]
\[
-\frac{kt_m}{m} = \ln \frac{1}{kv_0/mg + 1} \]
\[
t_m = \frac{m}{k} \ln \left( 1 + \frac{kv_0}{mg} \right) \]

The Taylor series expansion of \( \ln(1 + x) \) about \( x = 0 \) is
\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots, \]
so as long as \( kv_0/mg \) is really small \( (kv_0/mg \ll 1) \), \( t_m \) can be approximated well by
\[
t_m \approx \frac{m}{k} \left[ \frac{kv_0}{mg} - \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 + \frac{1}{3} \left( \frac{kv_0}{mg} \right)^3 - \frac{1}{4} \left( \frac{kv_0}{mg} \right)^4 + \cdots \right] \]
\[
\approx \frac{v_0}{g} \left[ 1 - \frac{1}{2} \left( \frac{kv_0}{mg} \right) + \frac{1}{3} \left( \frac{kv_0}{mg} \right)^2 - \frac{1}{4} \left( \frac{kv_0}{mg} \right)^3 + \cdots \right]. \]

To find the maximum height, plug \( t_m \) into the formula for \( y(t) \).
\[
y(t_m) = -\frac{mg}{k} \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) \left( 1 - e^{-kt_m/m} \right) \]
\[
= -\frac{mg}{k} \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) \left( 1 - \exp \left[ -\frac{k}{m} \ln \left( 1 + \frac{kv_0}{mg} \right) \right] \right) \]
\[
= -\frac{mg}{k^2} \ln \left( 1 + \frac{kv_0}{mg} \right) + \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) \left[ 1 - \exp \left( -\ln \left( \frac{mg + kv_0}{mg} \right) \right) \right] \]
\[
= -\frac{mg}{k^2} \ln \left( 1 + \frac{kv_0}{mg} \right) + \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) \left( 1 - \frac{mg}{mg + kv_0} \right) \]
\[
= -\frac{mg}{k^2} \ln \left( 1 + \frac{kv_0}{mg} \right) + \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) \left( 1 - \frac{kv_0}{mg + kv_0} \right) \]
\[
= -\frac{mg}{k^2} \ln \left( 1 + \frac{kv_0}{mg} \right) + \frac{mv_0}{k} \frac{mv_0}{mg} \frac{mv_0}{mg + kv_0} \]
\[
= -\frac{mg}{k^2} \ln \left( 1 + \frac{kv_0}{mg} \right) + \frac{mv_0}{k} \frac{mv_0}{mg + kv_0} \]

As long as \( kv_0/mg \) is really small, that is, \( kv_0/mg \ll 1 \), we can use the Taylor series expansion for \( \ln(1 + kv_0/mg) \) to approximate the height.
\[
y(t_m) \approx -\frac{mg}{k^2} \left[ \frac{kv_0}{mg} - \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 + \frac{1}{3} \left( \frac{kv_0}{mg} \right)^3 - \frac{1}{4} \left( \frac{kv_0}{mg} \right)^4 + \cdots \right] + \frac{mv_0}{k} \]
\[
\approx -\frac{mg}{k^2} \frac{kv_0}{mg} + \frac{mv_0}{k} + \frac{mg}{k^2} \left[ \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 - \frac{1}{3} \left( \frac{kv_0}{mg} \right)^3 + \frac{1}{4} \left( \frac{kv_0}{mg} \right)^4 - \cdots \right] \]
\[
\approx \frac{v_0^2}{2g} \left[ 1 - \frac{2}{3} \left( \frac{kv_0}{mg} \right) + \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 - \cdots \right] \]

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Part (c)

Because $kv$ is a force, the product of $k$ and $v$ must be in Newtons.

$$[kv] = \text{N}$$

$$[k] \frac{m}{s} = \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$$

$$[k] = \frac{\text{kg}}{s}$$

Now we can look at the units of the ratio.

$$\left[ \frac{kv_0}{mg} \right] = \frac{\frac{\text{kg}}{s} \cdot \frac{\text{m}}{s}}{\text{kg} \cdot \frac{\text{m}}{\text{s}^2}} = \frac{\text{kg} \cdot \frac{\text{m}}{\text{s}^2}}{\text{kg} \cdot \frac{\text{m}}{\text{s}^2}} = 1$$

Therefore, $kv_0/mg$ is dimensionless.