Problem 28

A mass of 0.25 kg is dropped from rest in a medium offering a resistance of 0.2|v|, where v is measured in m/s.

(a) If the mass is dropped from a height of 30 m, find its velocity when it hits the ground.

(b) If the mass is to attain a velocity of no more than 10 m/s, find the maximum height from which it can be dropped.

(c) Suppose that the resistive force is k|v|, where v is measured in m/s and k is a constant. If the mass is dropped from a height of 30 m and must hit the ground with a velocity of no more than 10 m/s, determine the coefficient of resistance k that is required.

Solution

According to Newton’s second law, the equation of motion for the mass is

\[ \sum F = ma. \]

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the y-direction is relevant.

\[ \sum F_y = ma_y \]

Part (a)

Draw the free-body diagram for the falling object, taking the positive y-axis to point upward.

The minus sign in front of 0.2v is due to the fact that the mass is moving down and +y points up. The equation of motion can now be written.

\[ -0.2v - mg = ma_y \]

Replace \( a_y \) with \( dv/dt \) and bring 0.2v to the other side.

\[ m \frac{dv}{dt} + 0.2v = -mg \]
Divide both sides by \( m \).
\[
\frac{dv}{dt} + \frac{0.2}{m} v = -g
\]
This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).
\[
I = \exp \left( \int \frac{0.2}{m} \, ds \right) = e^{0.2t/m}
\]
Proceed with the multiplication.
\[
e^{0.2t/m} \frac{dv}{dt} + \frac{0.2}{m} e^{0.2t/m} v = -ge^{0.2t/m}
\]
The left side can be written as \( d/dt (Iv) \) by the product rule.
\[
\frac{d}{dt} (e^{0.2t/m} v) = -ge^{0.2t/m}
\]
Integrate both sides with respect to \( t \).
\[
e^{0.2t/m} v = -5mg e^{0.2t/m} + C_1
\]
Divide both sides by \( e^{0.2t/m} \).
\[
v(t) = -5mg + C_1 e^{-0.2t/m}
\]
Since the mass is dropped from rest, the initial condition for \( v \) is \( v(0) = 0 \). Use it to determine \( C_1 \).
\[
v(0) = -5mg + C_1 = 0 \quad \rightarrow \quad C_1 = 5mg
\]
Therefore, the velocity is
\[
v(t) = -5mg + 5mg e^{-0.2t/m} = -5mg(1 - e^{-0.2t/m}).
\]
Replace \( v \) with \( dy/dt \) to get an ODE for the position.
\[
\frac{dy}{dt} = -5mg(1 - e^{-0.2t/m})
\]
Integrate both sides with respect to \( t \).
\[
y(t) = -5mgt - 25m^2 g e^{-0.2t/m} + C_2 \quad (1)
\]
Use the initial condition \( y(0) = 30 \) to determine \( C_2 \).
\[
y(0) = -25m^2 g + C_2 = 30 \quad \rightarrow \quad C_2 = 30 + 25m^2 g
\]
Therefore,
\[
y(t) = -5mgt - 25m^2 g e^{-0.2t/m} + 30 + 25m^2 g
\]
\[
= 5(6 - mgt) + 25m^2 g(1 - e^{-0.2t/m}).
\]
Solve \( y(t) = 0 \) for \( t \) now to find when the mass hits the floor.
\[
0 = 5(6 - mgt) + 25m^2 g(1 - e^{-0.2t/m})
\]
Plug in \( m = 0.25 \) and \( g = 9.81 \) and then graph the function on the right side versus \( t \). The curve crosses the \( t \)-axis in two places: \( t \approx -1.866 \) and \( t \approx 3.628 \). We require a positive number, so we choose the second time. Calculate \( v(3.628) \) to determine the velocity of the mass when it hits the floor.

\[
v(3.628) \approx -5mg(1 - e^{-0.2\cdot 3.628/m}) \approx -11.59 \text{ meters/s}
\]

**Part (b)**

Return to equation (1).

\[
y(t) = -5mgt - 25m^2ge^{-0.2t/m} + C_2
\]

Rather than \( y(0) = 30 \), apply the initial condition \( y(0) = y_0 \) to determine \( C_2 \).

\[
y(0) = -25m^2g + C_2 = y_0 \quad \rightarrow \quad C_2 = y_0 + 25m^2g
\]

As a result,

\[
y(t) = -5mgt - 25m^2ge^{-0.2t/m} + y_0 + 25m^2g
\]

\[
= y_0 - 5mgt + 25m^2g(1 - e^{-0.2t/m}).
\]

Set \( v(t) = -10, m = 0.25, \) and \( g = 9.81 \) and solve for \( t \).

\[
-10 = -5mg(1 - e^{-0.2t/m}) \quad \rightarrow \quad t \approx 2.113
\]

Now set \( y(t) = 0 \) and \( t \approx 2.113 \) and solve for \( y_0 \).

\[
0 = y_0 - 5mg(2.113) + 25m^2g[1 - e^{-0.2(2.113)/m}]
\]

\[
y_0 \approx 13.41 \text{ meters}
\]

This is how high the mass can be dropped from while having a velocity of \(-10 \) m/s when it hits the floor.

**Part (c)**

Use the same free-body diagram as before, but use \( k \) instead of 0.2.

\[
\begin{array}{c}
\text{Going Down} \\
y \\
\end{array}
\]

The minus sign in front of \( kv \) is due to the fact that the mass is moving down and +\( y \) points up. The new equation of motion can now be written.

\[
-kv - mg = m\ddot{y}
\]
Replace $a_y$ with $dv/dt$ and bring $kv$ to the other side.

$$m \frac{dv}{dt} + kv = -mg$$

Divide both sides by $m$.

$$\frac{dv}{dt} + \frac{k}{m} v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp \left( \int \frac{k}{m} ds \right) = e^{kt/m}$$

Proceed with the multiplication.

$$e^{kt/m} \frac{dv}{dt} + \frac{k}{m} e^{kt/m} v = -ge^{kt/m}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{kt/m} v) = -ge^{kt/m}$$

Integrate both sides with respect to $t$.

$$e^{kt/m} v = -\frac{mg}{k} e^{kt/m} + C_3$$

Divide both sides by $e^{kt/m}$.

$$v(t) = -\frac{mg}{k} + C_3 e^{-kt/m}$$

Since the mass is dropped from rest, the initial condition for $v$ is $v(0) = 0$. Use it to determine $C_3$.

$$v(0) = -\frac{mg}{k} + C_3 = 0 \quad \rightarrow \quad C_3 = \frac{mg}{k}$$

Therefore, the velocity is

$$v(t) = -\frac{mg}{k} + \frac{mg}{k} e^{-kt/m}$$

$$= -\frac{mg}{k} (1 - e^{-kt/m}).$$

Replace $v$ with $dy/dt$ to get an ODE for the position.

$$\frac{dy}{dt} = -\frac{mg}{k} + \frac{mg}{k} e^{-kt/m}$$

Integrate both sides with respect to $t$.

$$y(t) = -\frac{mg}{k} t - \frac{m^2 g}{k^2} e^{-kt/m} + C_4$$

Use the initial condition $y(0) = 30$ to determine $C_4$.

$$y(0) = -\frac{m^2 g}{k^2} + C_4 = 30 \quad \rightarrow \quad C_4 = 30 + \frac{m^2 g}{k^2}$$

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Therefore,

\[ y(t) = -\frac{mg}{k} t - \frac{m^2 g}{k^2} e^{-kt/m} + 30 + \frac{m^2 g}{k^2} \]

\[ = 30 - \frac{mg}{k} t + \frac{m^2 g}{k^2} (1 - e^{-kt/m}). \]

Now we set the constraints to determine \( k \): plug in \( m = 0.25 \) and \( g = 9.81 \) and then set \( v(t) = -10 \) and \( y(t) = 0 \) to obtain a system of two equations for \( k \) and \( t \).

\[ 0 = 30 - \frac{(0.25)(9.81)}{k} t + \frac{(0.25)^2(9.81)}{k^2} (1 - e^{-kt/(0.25)}) \]

\[ -10 = - \frac{(0.25)(9.81)}{k} (1 - e^{-kt/(0.25)}) \]

Plot these two curves in the \( tk \)-plane and note where they intersect to find the solution.

We see that the intersection occurs at \( k \approx 0.24 \) and \( t \approx 3.96 \). This means that the mass will hit the floor in 3.96 seconds if \( k \approx 0.24 \) kg/s. This is the lowest value of \( k \) that will result in a velocity of \(-10 \) m/s at the floor.

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