Problem 28

A mass of 0.25 kg is dropped from rest in a medium offering a resistance of $0.2|v|$, where v is measured in m/s.

(a) If the mass is dropped from a height of 30 m, find its velocity when it hits the ground.

(b) If the mass is to attain a velocity of no more than 10 m/s, find the maximum height from which it can be dropped.

(c) Suppose that the resistive force is $k|v|$, where v is measured in m/s and k is a constant. If the mass is dropped from a height of 30 m and must hit the ground with a velocity of no more than 10 m/s, determine the coefficient of resistance k that is required.

Solution

According to Newton's second law, the equation of motion for the mass is

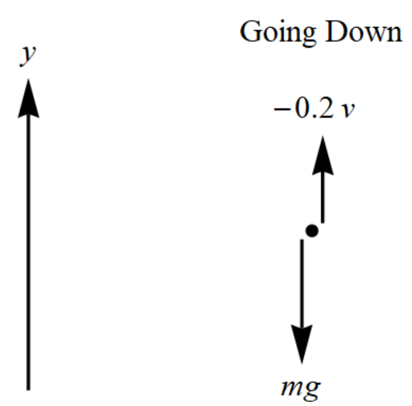
$$\sum \mathbf{F} = m\mathbf{a}.$$

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the y-direction is relevant.

$$\sum F_y = ma_y$$

Part (a)

Draw the free-body diagram for the falling object, taking the positive y-axis to point upward.



The minus sign in front of $0.2v$ is due to the fact that the mass is moving down and $+y$ points up. The equation of motion can now be written.

$$-0.2v - mg = ma_y$$

Replace a_y with dv/dt and bring $0.2v$ to the other side.

$$m\frac{dv}{dt} + 0.2v = -mg$$

Divide both sides by m.

$$\frac{dv}{dt} + \frac{0.2}{m}v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{0.2}{m}\,ds\right) = e^{0.2t/m}$$

Proceed with the multiplication.

$$e^{0.2t/m}\frac{dv}{dt} + \frac{0.2}{m}e^{0.2t/m}v = -ge^{0.2t/m}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{0.2t/m}v) = -ge^{0.2t/m}$$

Integrate both sides with respect to t.

$$e^{0.2t/m}v = -5mge^{0.2t/m} + C_1$$

Divide both sides by $e^{0.2t/m}$.

$$v(t) = -5mg + C_1e^{-0.2t/m}$$

Since the mass is dropped from rest, the initial condition for v is $v(0) = 0$. Use it to determine C_1.

$$v(0) = -5mg + C_1 = 0 \quad \rightarrow \quad C_1 = 5mg$$

Therefore, the velocity is

$$\begin{aligned} v(t) &= -5mg + 5mge^{-0.2t/m} \\ &= -5mg(1 - e^{-0.2t/m}). \end{aligned}$$

Replace v with dy/dt to get an ODE for the position.

$$\frac{dy}{dt} = -5mg(1 - e^{-0.2t/m})$$

Integrate both sides with respect to t.

$$y(t) = -5mgt - 25m^2ge^{-0.2t/m} + C_2 \tag{1}$$

Use the initial condition $y(0) = 30$ to determine C_2.

$$y(0) = -25m^2g + C_2 = 30 \quad \rightarrow \quad C_2 = 30 + 25m^2g$$

Therefore,

$$\begin{aligned} y(t) &= -5mgt - 25m^2ge^{-0.2t/m} + 30 + 25m^2g \\ &= 5(6 - mgt) + 25m^2g(1 - e^{-0.2t/m}). \end{aligned}$$

Solve $y(t) = 0$ for t now to find when the mass hits the floor.

$$0 = 5(6 - mgt) + 25m^2g(1 - e^{-0.2t/m})$$

Plug in $m = 0.25$ and $g = 9.81$ and then graph the function on the right side versus t. The curve crosses the t-axis in two places: $t \approx -1.866$ and $t \approx 3.628$. We require a positive number, so we choose the second time. Calculate $v(3.628)$ to determine the velocity of the mass when it hits the floor.

$$v(3.628) \approx -5mg(1 - e^{-0.2 \cdot 3.628/m}) \approx -11.59 \, \frac{\text{meters}}{\text{s}}$$

Part (b)

Return to equation (1).

$$y(t) = -5mgt - 25m^2 g e^{-0.2t/m} + C_2 \tag{1}$$

Rather than $y(0) = 30$, apply the initial condition $y(0) = y_0$ to determine C_2.

$$y(0) = -25m^2 g + C_2 = y_0 \quad \rightarrow \quad C_2 = y_0 + 25m^2 g$$

As a result,

$$\begin{aligned} y(t) &= -5mgt - 25m^2 g e^{-0.2t/m} + y_0 + 25m^2 g \\ &= y_0 - 5mgt + 25m^2 g(1 - e^{-0.2t/m}). \end{aligned}$$

Set $v(t) = -10$, $m = 0.25$, and $g = 9.81$ and solve for t.

$$-10 = -5mg(1 - e^{-0.2t/m}) \quad \rightarrow \quad t \approx 2.113$$

Now set $y(t) = 0$ and $t \approx 2.113$ and solve for y_0.

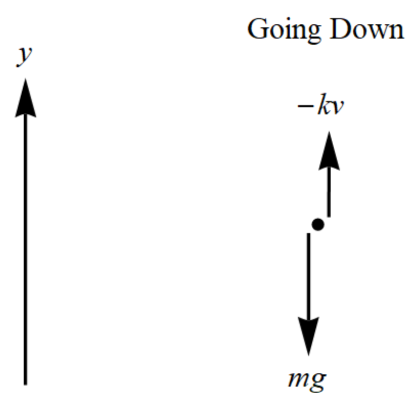
$$0 = y_0 - 5mg(2.113) + 25m^2 g[1 - e^{-0.2(2.113)/m}]$$

$$y_0 \approx 13.41 \text{ meters}$$

This is how high the mass can be dropped from while having a velocity of -10 m/s when it hits the floor.

Part (c)

Use the same free-body diagram as before, but use k instead of 0.2.



The minus sign in front of kv is due to the fact that the mass is moving down and $+y$ points up. The new equation of motion can now be written.

$$-kv - mg = ma_y$$

Replace a_y with dv/dt and bring kv to the other side.

$$m\frac{dv}{dt} + kv = -mg$$

Divide both sides by m.

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{k}{m}\,ds\right) = e^{kt/m}$$

Proceed with the multiplication.

$$e^{kt/m}\frac{dv}{dt} + \frac{k}{m}e^{kt/m}v = -ge^{kt/m}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{kt/m}v) = -ge^{kt/m}$$

Integrate both sides with respect to t.

$$e^{kt/m}v = -\frac{mg}{k}e^{kt/m} + C_3$$

Divide both sides by $e^{kt/m}$.

$$v(t) = -\frac{mg}{k} + C_3e^{-kt/m}$$

Since the mass is dropped from rest, the initial condition for v is $v(0) = 0$. Use it to determine C_3.

$$v(0) = -\frac{mg}{k} + C_3 = 0 \quad \rightarrow \quad C_3 = \frac{mg}{k}$$

Therefore, the velocity is

$$\begin{aligned} v(t) &= -\frac{mg}{k} + \frac{mg}{k}e^{-kt/m} \\ &= -\frac{mg}{k}(1 - e^{-kt/m}). \end{aligned}$$

Replace v with dy/dt to get an ODE for the position.

$$\frac{dy}{dt} = -\frac{mg}{k} + \frac{mg}{k}e^{-kt/m}$$

Integrate both sides with respect to t.

$$y(t) = -\frac{mg}{k}t - \frac{m^2g}{k^2}e^{-kt/m} + C_4$$

Use the initial condition $y(0) = 30$ to determine C_4.

$$y(0) = -\frac{m^2g}{k^2} + C_4 = 30 \quad \rightarrow \quad C_4 = 30 + \frac{m^2g}{k^2}$$

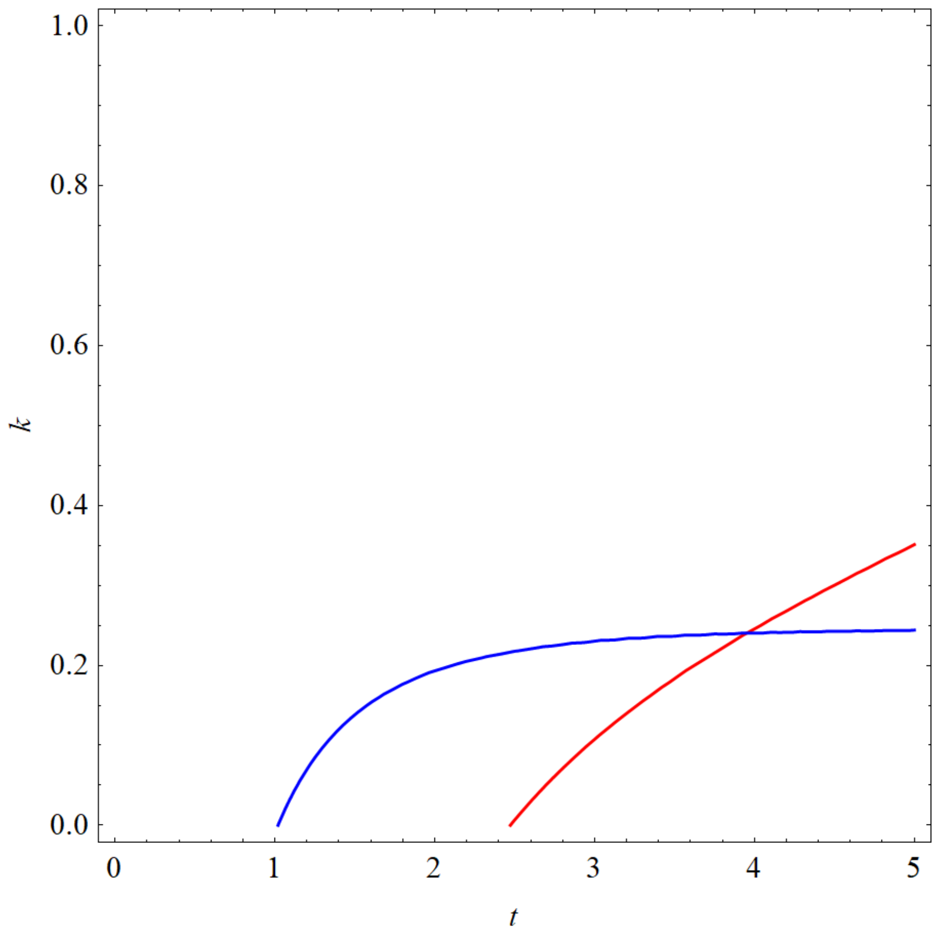
Therefore,

$$y(t) = -\frac{mg}{k}t - \frac{m^2g}{k^2}e^{-kt/m} + 30 + \frac{m^2g}{k^2}$$
$$= 30 - \frac{mg}{k}t + \frac{m^2g}{k^2}(1 - e^{-kt/m}).$$

Now we set the constraints to determine k: plug in $m = 0.25$ and $g = 9.81$ and then set $v(t) = -10$ and $y(t) = 0$ to obtain a system of two equations for k and t.

$$0 = 30 - \frac{(0.25)(9.81)}{k}t + \frac{(0.25)^2(9.81)}{k^2}(1 - e^{-kt/(0.25)})$$
$$-10 = -\frac{(0.25)(9.81)}{k}(1 - e^{-kt/(0.25)})$$

Plot these two curves in the tk-plane and note where they intersect to find the solution.



We see that the intersection occurs at $k \approx 0.24$ and $t \approx 3.96$. This means that the mass will hit the floor in 3.96 seconds if $k \approx 0.24$ kg/s. This is the lowest value of k that will result in a velocity of -10 m/s at the floor.