Problem 30

Let \( v(t) \) and \( w(t) \) be the horizontal and vertical components, respectively, of the velocity of a batted (or thrown) baseball. In the absence of air resistance, \( v \) and \( w \) satisfy the equations

\[
\frac{dv}{dt} = 0, \quad \frac{dw}{dt} = -g.
\]

(a) Show that

\[
v = u \cos A, \quad w = -gt + u \sin A,
\]

where \( u \) is the initial speed of the ball and \( A \) is its initial angle of elevation.

(b) Let \( x(t) \) and \( y(t) \) be the horizontal and vertical coordinates, respectively, of the ball at time \( t \). If \( x(0) = 0 \) and \( y(0) = h \), find \( x(t) \) and \( y(t) \) at any time \( t \).

(c) Let \( g = 32 \text{ ft/s}^2 \), \( u = 125 \text{ ft/s} \), and \( h = 3 \text{ ft} \). Plot the trajectory of the ball for several values of the angle \( A \); that is, plot \( x(t) \) and \( y(t) \) parametrically.

(d) Suppose the outfield wall is at a distance \( L \) and has height \( H \). Find a relation between \( u \) and \( A \) that must be satisfied if the ball is to clear the wall.

(e) Suppose that \( L = 350 \text{ ft} \) and \( H = 10 \text{ ft} \). Using the relation in part (d), find (or estimate from a plot) the range of values of \( A \) that correspond to an initial velocity of \( u = 110 \text{ ft/s} \).

(f) For \( L = 350 \text{ and } H = 10 \), find the minimum initial velocity \( u \) and the corresponding optimal angle \( A \) for which the ball will clear the wall.

[TYPO: These should read \( L = 350 \text{ ft} \) and \( H = 10 \text{ ft} \).]

Solution

Below is a figure of the baseball with initial speed \( u \) at an angle \( A \).

\[\text{Part (a)}\]

Integrate both ODEs with respect to \( t \).

\[
\frac{dv}{dt} = 0 \quad \frac{dw}{dt} = -g
\]

\[
v(t) = C_1 \quad w(t) = -gt + C_2
\]

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Decompose the initial velocity vector into its components along the \( x \) - and \( y \)-axes.

Use these components to determine \( C_1 \) and \( C_2 \).

\[
\begin{align*}
v(0) &= C_1 = u \cos A \\
w(0) &= C_2 = u \sin A
\end{align*}
\]

Therefore,

\[
\begin{align*}
v(t) &= u \cos A \\
w(t) &= -gt + u \sin A.
\end{align*}
\]

**Part (b)**

Integrate the velocities to get the positions.

\[
\begin{align*}
x(t) &= \int v(t) \, dt = (u \cos A) t + C_3 \\
y(t) &= \int w(t) \, dt = -\frac{1}{2} gt^2 + (u \sin A) t + C_4
\end{align*}
\]

Use the initial conditions, \( x(0) = 0 \) and \( y(0) = h \), to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
x(0) &= C_3 = 0 \\
y(0) &= C_4 = h
\end{align*}
\]

Therefore,

\[
\begin{align*}
x(t) &= (u \cos A) t \\
y(t) &= -\frac{1}{2} gt^2 + (u \sin A) t + h.
\end{align*}
\]

**Part (c)**

Let \( g = 32 \) \( \text{ft/s}^2 \), \( u = 125 \) \( \text{ft/s} \), and \( h = 3 \) \( \text{ft} \).

\[
\begin{align*}
x(t) &= (125 \cos A) t \\
y(t) &= -16t^2 + (125 \sin A) t + 3.
\end{align*}
\]
Below are graphs for $A = 10^\circ$, $A = 20^\circ$, $A = 30^\circ$, $A = 40^\circ$, $A = 50^\circ$, and $A = 60^\circ$ in purple, blue, green, yellow, orange, and red, respectively.

**Part (d)**

Set $x(t) = L$ and $y(t) \geq H$.

\[
L = (u \cos A)t
\]

\[
H \leq -\frac{1}{2}gt^2 + (u \sin A)t + h
\]

Since we just want a formula involving $u$ and $A$, we will eliminate $t$. Solve the first equation for it.

\[
t = \frac{L}{u \cos A}
\]

Substitute this result into the inequality.

\[
H \leq -\frac{1}{2}g \left( \frac{L}{u \cos A} \right)^2 + (u \sin A) \left( \frac{L}{u \cos A} \right) + h
\]

Therefore,

\[
H \leq -\frac{gL^2}{2u^2} \sec^2 A + L \tan A + h.
\]

**Part (e)**

Set $H = 10$ ft, $L = 350$ ft, $g = 32$ ft/s$^2$, $u = 110$ ft/s, and $h = 3$ ft in the result of part (d).

\[
10 \leq -\frac{(32)(350)^2}{2(110)^2} \sec^2 A + 350 \tan A + 3
\]

\[
-\frac{(32)(350)^2}{2(110)^2} (\tan^2 A + 1) + 350 \tan A - 7 \geq 0
\]

\[-\frac{19600}{121} \tan^2 A + 350 \tan A - \frac{20447}{121} \geq 0
\]
Below is a graph of the function on the left side versus \( \tan A \). We care about the part of the parabola that lies above the horizontal axis.

Use the quadratic formula to locate the zeros.

\[
-350 + \sqrt{350^2 - 4 \left( \frac{19600}{121} \right) \left( \frac{20447}{121} \right)} \leq \tan A \leq -350 - \sqrt{350^2 - 4 \left( \frac{19600}{121} \right) \left( \frac{20447}{121} \right)} \\
0.7283 \lesssim \tan A \lesssim 1.4324 \\
\tan^{-1} 0.7283 \lesssim A \lesssim \tan^{-1} 1.4324
\]

Therefore, in radians

\[0.6295 \lesssim A \lesssim 0.9613,\]

or in degrees

\[36.07^\circ \lesssim A \lesssim 55.08^\circ.\]

**Part (f)**

From the result of part (b), the position of the baseball is given by

\[x(t) = (u \cos A)t\]
\[y(t) = -\frac{1}{2} gt^2 + (u \sin A)t + h.\]

Set \( x(t) = L \) and \( y(t) = H \).

\[L = (u \cos A)t\]
\[H = -\frac{1}{2} gt^2 + (u \sin A)t + h.\]

Solve the first equation for \( t \)

\[t = \frac{L}{u \cos A}\]

and then substitute it into the second equation.

\[H = -\frac{1}{2} g \left( \frac{L}{u \cos A} \right)^2 + (u \sin A) \left( \frac{L}{u \cos A} \right) + h\]
\[= -\frac{gL^2}{2u^2 \cos^2 A} + L \tan A + h\]
Solve for \( u \).

\[
\frac{gL^2}{2u^2 \cos^2 A} = L \tan A + h - H
\]

\[
\frac{gL^2}{2u^2} = L \tan A \cos^2 A + (h - H) \cos^2 A
\]

\[
\frac{1}{u^2} = \frac{2}{gL^2} [L \sin A \cos A + (h - H) \cos^2 A]
\]

\[
u = \sqrt{\frac{gL^2}{2L \sin A \cos A + (h - H) \cos^2 A}}
\]

\[
u = \sqrt{\frac{gL^2}{2L \sin A \cos A + 2(h - H) \cos^2 A}}
\]

\[
u = \sqrt{\frac{gL^2}{L \sin 2A + 2(h - H) \cos^2 A}}
\]

Plug in \( g = 32 \text{ ft/s}^2 \), \( L = 350 \text{ ft} \), \( H = 10 \text{ ft} \), and \( h = 3 \text{ ft} \).

\[
u = 200\sqrt{7}(25 \sin 2A - \cos^2 A)^{-1/2}
\]  \hspace{1cm} (1)

Differentiate \( u \) with respect to \( A \) and then set it equal to zero to find the values of \( A \) that extremize \( u \).

\[
\frac{du}{dA} = -100\sqrt{7}(25 \sin 2A - \cos^2 A)^{-3/2}[50 \cos 2A - 2 \cos A(- \sin A)] = 0
\]

\[
-100\sqrt{7}(25 \sin 2A - \cos^2 A)^{-3/2}(50 \cos 2A + 2 \sin A \cos A) = 0
\]

\[
50 \cos 2A + 2 \sin A \cos A = 0
\]

\[
50 \cos 2A + \sin 2A = 0
\]

\[
50 \cos 2A = - \sin 2A
\]

\[
2500 \cos^2 2A = \sin^2 2A
\]

\[
2500 \cos^2 2A = 1 - \cos^2 2A
\]

\[
2501 \cos^2 2A = 1
\]

\[
\cos^2 2A = \frac{1}{2501}
\]

\[
\cos 2A = \pm \frac{1}{\sqrt{2501}}
\]

\[
2A = \left\{ \pm \cos^{-1} \frac{1}{\sqrt{2501}}, \pm \cos^{-1} \frac{-1}{\sqrt{2501}} \right\}
\]

The two minus signs can be discarded because they don’t result in values of \( A \) between 0 and 90°.

\[
2A = \left\{ \cos^{-1} \frac{1}{\sqrt{2501}}, \cos^{-1} \frac{-1}{\sqrt{2501}} \right\}
\]
As a result, the optimal angles that extremize $u$ are

$$A = \left\{ \frac{1}{2} \cos^{-1} \frac{1}{\sqrt{2501}}, \frac{1}{2} \cos^{-1} \frac{-1}{\sqrt{2501}} \right\} \approx \{0.775399, 0.795397\}.$$

Plug these values of $A$ into equation (1) to find out the initial speeds associated with these angles.

$$u(A \approx 0.775399) \approx 106.937$$
$$u(A \approx 0.795397) \approx 106.894$$

Therefore, the minimum initial speed that a baseball can have to get to $(L, H)$ is about 106.894 ft/s, and the angle from the horizontal it has to have is 0.795397 radians, or 45.57°.