Problem 9

In each of Problems 7 through 12, state where in the $ty$-plane the hypotheses of Theorem 2.4.2 are satisfied.

$$y' = \frac{\ln |ty|}{1 - t^2 + y^2}$$

Solution

According to Theorem 2.4.2, a unique solution to

$$y' = f(t, y), \quad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that $f$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $\alpha < t < \beta, \gamma < y < \delta$ that contains $(t_0, y_0)$. In this exercise

$$f(t, y) = \frac{\ln ty}{1 - t^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{ty} \cdot t(1 - t^2 + y^2) - (2y) \ln ty = \frac{1 - t^2 + y^2 - 2y^2 \ln ty}{y(1 - t^2 + y^2)^2}$$

in the first and third quadrants of the $ty$-plane and

$$f(t, y) = \frac{\ln (ty)}{1 - t^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{ty} \cdot (-t)(1 - t^2 + y^2) - (2y) \ln (ty) = \frac{1 - t^2 + y^2 - 2y^2 \ln (-ty)}{y(1 - t^2 + y^2)^2}$$

in the second and fourth quadrants of the $ty$-plane. $f$ is continuous as long as $|ty| > 0$ and $1 - t^2 + y^2 \neq 0$, and $\frac{\partial f}{\partial y}$ is continuous as long as $|ty| > 0$ and $1 - t^2 + y^2 \neq 0$. Therefore, the hypotheses of Theorem 2.4.2 are satisfied if $t \neq 0$ and $y \neq 0$ and $(1 - t^2 + y^2 < 0$ or $1 - t^2 + y^2 > 0)$.