Problem 13

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value $y_0$.

$$y' = -\frac{4t}{y}, \quad y(0) = y_0$$

Solution

Method Using the Chain Rule

$$\frac{dy}{dt} = -\frac{4t}{y}$$

Multiply both sides by $2y$.

$$2y \frac{dy}{dt} = -8t$$

The left side can be written as $d/dt(y^2)$ by the chain rule.

$$\frac{d}{dt}(y^2) = -8t$$

Integrate both sides with respect to $t$.

$$y^2 = -4t^2 + C_1$$

Apply the initial condition now to determine $C_1$.

$$y_0^2 = C_1$$

So then the previous equation becomes

$$y^2 = -4t^2 + y_0^2.$$ 

Therefore,

$$y(t) = \pm \sqrt{y_0^2 - 4t^2}.$$
Method By Separating Variables

\[ \frac{dy}{dt} = -\frac{4t}{y} \]

Solve the ODE by separating variables.

\[ y \, dy = -4t \, dt \]

Integrate both sides.

\[ \frac{1}{2} y^2 = -2t^2 + C_2 \]

Multiply both sides by 2.

\[ y^2 = -4t^2 + 2C_2 \]

Apply the initial condition now to determine \( C_2 \).

\[ y_0^2 = 2C_2 \]

So then the previous equation becomes

\[ y^2 = -4t^2 + y_0^2. \]

Therefore,

\[ y(t) = \pm \sqrt{y_0^2 - 4t^2}. \]

According to Theorem 2.4.2, a unique solution to

\[ y' = f(t, y), \quad y(t_0) = y_0 \]

exists in some interval \( t_0 - h < t < t_0 + h \) within \( \alpha < t < \beta \), provided that \( f \) and \( \frac{\partial f}{\partial y} \) are continuous in a rectangle \( \alpha < t < \beta, \gamma < y < \delta \) that contains \( (t_0, y_0) \). In this exercise

\[ f(t, y) = -\frac{4t}{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{4t}{y^2}. \]

Both \( f \) and \( \frac{\partial f}{\partial y} \) are discontinuous at \( y = 0 \), so as long as \( y_0 \neq 0 \), there will be an interval of \( t \) where a unique solution exists.