Problem 14

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value $y_0$.

$$y' = 2ty^2, \quad y(0) = y_0$$

Solution

Method Using the Chain Rule

$$y' = 2ty^2$$

Divide both sides by $y^2$.

$$\frac{y'}{y^2} = 2t$$

The left side can be written as $d/dt(-1/y)$ by the chain rule.

$$\frac{d}{dt} \left(-\frac{1}{y}\right) = 2t$$

Integrate both sides with respect to $t$.

$$-\frac{1}{y} = t^2 + C_1$$

Apply the initial condition now to determine $C_1$.

$$-\frac{1}{y_0} = C_1$$

So then the previous equation becomes

$$-\frac{1}{y} = t^2 - \frac{1}{y_0}.$$ 

Therefore,

$$y(t) = -\frac{1}{t^2 - \frac{1}{y_0}}$$

$$= \frac{1}{y_0 - t^2}$$

$$= \frac{y_0}{1 - y_0t^2}.$$ 

Note that the solution blows up in a finite amount of time if $y_0$ is positive, specifically

$$1 - y_0t^2 = 0 \quad \rightarrow \quad t = \pm \sqrt{-\frac{1}{y_0}}.$$ 

No such thing occurs, though, if $y_0$ is not positive.
Method By Separating Variables

\[ \frac{dy}{dt} = 2ty^2 \]

Solve the ODE by separating variables.

\[ \frac{dy}{y^2} = 2t \, dt \]

Integrate both sides.

\[ -\frac{1}{y} = t^2 + C_2 \]

Apply the initial condition now to determine \( C_2 \).

\[ -\frac{1}{y_0} = C_2 \]

So then the previous equation becomes

\[ -\frac{1}{y} = t^2 - \frac{1}{y_0} \]

Therefore,

\[ y(t) = -\frac{1}{t^2 - \frac{1}{y_0}} \]

\[ = \frac{1}{y_0 - t^2} \]

\[ = \frac{y_0}{1 - y_0 t^2} \]

According to Theorem 2.4.2, a unique solution to

\[ y' = f(t, y), \quad y(t_0) = y_0 \]

exists in some interval \( t_0 - h < t < t_0 + h \) within \( \alpha < t < \beta \), provided that \( f \) and \( \partial f / \partial y \) are continuous in a rectangle \( \alpha < t < \beta, \gamma < y < \delta \) that contains \((t_0, y_0)\). In this exercise

\[ f(t, y) = 2ty^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4ty. \]

Both \( f \) and \( \partial f / \partial y \) are continuous everywhere except where the solution for \( y \) blows up. If \( y_0 > 0 \), then a unique solution exists in an interval within

\[ -\sqrt{\frac{1}{y_0}} < t < \sqrt{\frac{1}{y_0}}, \]

and if \( y_0 \leq 0 \), then a unique solution exists in an interval within

\[ -\infty < t < \infty. \]