Problem 15

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value \( y_0 \).

\[
y' + y^3 = 0, \quad y(0) = y_0
\]

Solution

Method Using the Chain Rule

Bring \( y^3 \) to the right side.

\[
y' = -y^3
\]

Divide both sides by \( y^3 \).

\[
\frac{y'}{y^3} = -1
\]

The left side can be written as \( d/dt[-1/(2y^2)] \) by the chain rule.

\[
\frac{d}{dt} \left( -\frac{1}{2y^2} \right) = -1
\]

Integrate both sides with respect to \( t \).

\[
-\frac{1}{2y^2} = -t + C_1
\]

Apply the initial condition now to determine \( C_1 \).

\[
-\frac{1}{2y_0^2} = C_1
\]

So then the previous equation becomes

\[
-\frac{1}{2y^2} = -t - \frac{1}{2y_0^2}
\]

\[
y^2 = 2t + \frac{1}{y_0^2}
\]

Therefore, taking the square root of both sides,

\[
y(t) = \frac{y_0}{\sqrt{2ty_0^2 + 1}}
\]

The positive root was chosen so that the initial condition remains satisfied. Note that the solution blows up in a finite amount of time if

\[
2ty_0^2 + 1 = 0 \quad \rightarrow \quad t = -\frac{1}{2y_0^2}.
\]
Method By Separating Variables

Bring $y^3$ to the right side.

$$\frac{dy}{dt} = -y^3$$

Separate variables.

$$\frac{dy}{y^3} = -dt$$

Integrate both sides.

$$-\frac{1}{2y^2} = t + C_2$$

Apply the initial condition now to determine $C_2$.

$$-\frac{1}{2y_0^2} = C_2$$

So then the previous equation becomes

$$-\frac{1}{2y^2} = t - \frac{1}{2y_0^2}$$

$$\frac{1}{y^2} = 2t + \frac{1}{y_0^2}$$

$$y^2 = \frac{1}{2t + \frac{1}{y_0^2}}$$

$$= \frac{y_0^2}{2ty_0^2 + 1}.$$  

Therefore, taking the square root of both sides,

$$y(t) = \frac{y_0}{\sqrt{2ty_0^2 + 1}}.$$  

The positive root was chosen so that the initial condition remains satisfied. According to Theorem 2.4.2, a unique solution to

$$y' = f(t, y), \quad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that $f$ and $\partial f/\partial y$ are continuous in a rectangle $\alpha < t < \beta, \gamma < y < \delta$ that contains $(t_0, y_0)$. In this exercise

$$f(t, y) = -y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = -3y^2.$$  

Both $f$ and $\partial f/\partial y$ are continuous everywhere except where the solution for $y$ blows up. Since the initial condition is given at $t = 0$, a unique solution exists in an interval within

$$-\frac{1}{2y_0^2} < t < \infty.$$