Problem 16

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' = t^2/y(1+t^3), \qquad y(0) = y_0$$

Solution

Method Using the Chain Rule

$$y' = \frac{t^2}{y(1+t^3)}$$

Multiply both sides by $2y$.

$$2yy' = \frac{2t^2}{1+t^3}$$

The left side can be written as $d/dt(y^2)$ by the chain rule.

$$\frac{d}{dt}(y^2) = \frac{2t^2}{1+t^3}$$

Integrate both sides with respect to t.

$$y^2 = \int^t \frac{2s^2}{1+s^3}\,ds$$

Evaluate the integral using the following substitution.

$$u = 1+s^3$$
$$du = 3s^2\,ds \quad \rightarrow \quad \frac{2}{3}\,du = 2s^2\,ds$$

As a result,

$$\begin{aligned} y^2 &= \int^{1+t^3} \frac{\frac{2}{3}\,du}{u} \\ &= \frac{2}{3}\ln|u|\bigg|^{1+t^3} \\ &= \frac{2}{3}\ln|1+t^3| + C_1. \end{aligned}$$

Apply the initial condition now to determine C_1.

$$y_0^2 = C_1$$

So then the previous equation becomes

$$y^2 = \frac{2}{3}\ln|1+t^3| + y_0^2.$$

Therefore,

$$y(t) = \pm\sqrt{\frac{2}{3}\ln|1+t^3| + y_0^2}.$$

Method By Separating Variables

$$\frac{dy}{dt} = \frac{t^2}{y(1+t^3)}$$

Separate variables.

$$y\,dy = \frac{t^2}{1+t^3}\,dt$$

Integrate both sides.

$$\frac{1}{2}y^2 = \int^t \frac{s^2}{1+s^3}\,ds$$

Multiply both sides by 2.

$$y^2 = \int^t \frac{2s^2}{1+s^3}\,ds$$

This is the same result as before, so the solution is the same.

$$y(t) = \pm\sqrt{\frac{2}{3}\ln|1+t^3| + y_0^2}.$$

According to Theorem 2.4.2, a unique solution to

$$y' = f(t,y), \qquad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that f and $\partial f/\partial y$ are continuous in a rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ that contains (t_0, y_0). In this exercise

$$f(t,y) = \frac{t^2}{y(1+t^3)} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{t^2}{y^2(1+t^3)}.$$

Both f and $\partial f/\partial y$ are discontinuous at $y = 0$. The times that corresponds to $y = 0$ are

$$\frac{2}{3}\ln|1+t^3| + y_0^2 = 0$$

$$\ln|1+t^3| = -\frac{3y_0^2}{2}$$

$$|1+t^3| = \exp\left(-\frac{3y_0^2}{2}\right)$$

$$1+t^3 = \pm\exp\left(-\frac{3y_0^2}{2}\right)$$

$$t = \sqrt[3]{\pm\exp\left(-\frac{3y_0^2}{2}\right) - 1}.$$

The plus and minus signs result in negative times, but the plus sign gives one smaller in magnitude. Since the initial condition is given at $t = 0$, a unique solution exists in an interval within

$$\sqrt[3]{\exp\left(-\frac{3y_0^2}{2}\right) - 1} < t < \infty.$$