Problem 21

Consider the initial value problem \( y' = y^{1/3}, \ y(0) = 0 \) from Example 3 in the text.

(a) Is there a solution that passes through the point \((1, 1)\)? If so, find it.

(b) Is there a solution that passes through the point \((2, 1)\)? If so, find it.

(c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at \( t = 2 \).

Solution

Suppose first that \( y \neq 0 \). Divide both sides of the ODE by \( y^{1/3} \).

\[
\frac{y'}{y^{1/3}} = 1
\]

The left side can be written as \( d/dt[(3/2)y^{2/3}] \) by using the chain rule.

\[
\frac{d}{dt} \left( \frac{3}{2} y^{2/3} \right) = 1
\]

Integrate both sides with respect to \( t \) to obtain the general solution.

\[
\frac{3}{2} y^{2/3} = t + C
\] (1)

Apply the initial condition \( y(0) = 0 \) to determine \( C \).

\[
0 = C
\]

So then the previous equation becomes

\[
\frac{3}{2} y^{2/3} = t
\]

\[
y^{2/3} = \frac{2t}{3}
\]

\[
y^2 = \left( \frac{2t}{3} \right)^3.
\]

Therefore, two nonzero solutions that pass through \((0, 0)\) are

\[
y(t) = \pm \left( \frac{2t}{3} \right)^{3/2}.
\]

By inspection, we see that \( y(t) = 0 \) also satisfies the initial value problem. The set of values at \( t = 2 \) are

\[
y(2) = \left\{ 0, \pm \left( \frac{4}{3} \right)^{3/2} \right\} \approx \{0, \pm 1.5396\}.\]
Part (a)

Rather than $y(0) = 0$, apply the initial condition $y(1) = 1$ to find $C$ instead.

$$\frac{3}{2} = 1 + C \rightarrow C = \frac{1}{2}$$

So then equation (1) becomes

$$\frac{3}{2} y^{2/3} = t + \frac{1}{2}$$

$$y^{2/3} = \frac{2}{3} t + \frac{1}{3}$$

$$y^2 = \left( \frac{2}{3} t + \frac{1}{3} \right)^3$$

$$y(t) = \pm \left( \frac{2}{3} t + \frac{1}{3} \right)^{3/2}.$$

Therefore, the solution that passes through $(1, 1)$ is

$$y(t) = \left( \frac{2}{3} t + \frac{1}{3} \right)^{3/2}.$$

Part (b)

Rather than $y(0) = 0$, apply the initial condition $y(2) = 1$ to find $C$ instead.

$$\frac{3}{2} = 2 + C \rightarrow C = -\frac{1}{2}$$

So then equation (1) becomes

$$\frac{3}{2} y^{2/3} = t - \frac{1}{2}$$

$$y^{2/3} = \frac{2}{3} t - \frac{1}{3}$$

$$y^2 = \left( \frac{2}{3} t - \frac{1}{3} \right)^3$$

$$y(t) = \pm \left( \frac{2}{3} t - \frac{1}{3} \right)^{3/2}.$$ 

Therefore, the solution that passes through $(2, 1)$ is

$$y(t) = \left( \frac{2}{3} t - \frac{1}{3} \right)^{3/2}.$$