Problem 27

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

\[ y' + p(t)y = q(t)y^n, \]

and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

(a) Solve Bernoulli’s equation when \( n = 0 \); when \( n = 1 \).

(b) Show that if \( n \neq 0, 1 \), then the substitution \( v = y^{1-n} \) reduces Bernoulli’s equation to a linear equation. This method of solution was found by Leibniz in 1696.

Solution

\( n = 0 \)

If \( n = 0 \), then Bernoulli’s equation reduces to

\[ y' + p(t)y = q(t), \]

which can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int p(s) \, ds \right) \]

Proceed with the multiplication.

\[ \exp \left( \int^t p(s) \, ds \right) y' + p(t) \exp \left( \int^t p(s) \, ds \right) y = q(t) \exp \left( \int^t p(s) \, ds \right) \]

The left side can be written as \( d/dt(ITY) \) by the product rule.

\[ \frac{d}{dt} \left[ \exp \left( \int^t p(s) \, ds \right) y \right] = q(t) \exp \left( \int^t p(s) \, ds \right) \]

Integrate both sides with respect to \( t \).

\[ \exp \left( \int^t p(s) \, ds \right) y = \int^t q(r) \exp \left( \int^r p(s) \, ds \right) \, dr + C_1 \]

Divide both sides by \( e^{\int^t p(s) \, ds} \) to solve for \( y \).

\[ y(t) = \exp \left( -\int^t p(s) \, ds \right) \int^t q(r) \exp \left( \int^r p(s) \, ds \right) \, dr + C_1 \exp \left( -\int^t p(s) \, ds \right) \]
If \( n = 1 \), then Bernoulli’s equation reduces to

\[ y' + p(t)y = q(t)y, \]

which can also be solved by multiplying both sides by an integrating factor \( I \). First, bring \( q(t)y \) to the left side and factor \( y \).

\[ y' + [p(t) - q(t)]y = 0 \]

Use the following integrating factor.

\[ I = \exp \left( \int [p(r) - q(r)] \, dr \right) \]

Proceed with the multiplication.

\[ \exp \left( \int [p(r) - q(r)] \, dr \right) y' + [p(t) - q(t)] \exp \left( \int [p(r) - q(r)] \, dr \right) y = 0 \]

The left side can be written as \( d/dt(Iy) \) by the product rule.

\[ \frac{d}{dt} \left[ \exp \left( \int [p(r) - q(r)] \, dr \right) y \right] = 0 \]

Integrate both sides with respect to \( t \).

\[ \exp \left( \int [p(r) - q(r)] \, dr \right) y = C_2 \]

Divide both sides by the exponential function to solve for \( y \).

\[ y(t) = C_2 \exp \left( - \int [p(r) - q(r)] \, dr \right) \]

If \( n \neq 0, 1 \),

\[ y' + p(t)y = q(t)y^n \]

Divide both sides by \( y^n \).

\[ y^{-n}y' + p(t)y^{1-n} = q(t) \]  \( (1) \)

Make the substitution \( u = y^{1-n} \). We now have to find what \( y' \) is in terms of this new variable. Differentiate both sides of the substitution with respect to \( t \), using the chain rule on the right side.

\[ \frac{du}{dt} = (1 - n)y^{-n} \cdot \frac{dy}{dt} \]

Divide both sides by \( 1 - n \).

\[ \frac{1}{1 - n} \frac{du}{dt} = y^{-n}y' \]

Substitute this result and \( u = y^{1-n} \) into equation (1) to obtain a linear ODE for \( u \).

\[ \frac{1}{1 - n} \frac{du}{dt} + p(t)u = q(t) \]