Problem 27

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

(a) Solve Bernoulli's equation when $n = 0$; when $n = 1$.

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

Solution

$\underline{\boldsymbol{n = 0}}$

If $n = 0$, then Bernoulli's equation reduces to

$$y' + p(t)y = q(t),$$

which can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t p(s)\,ds\right)$$

Proceed with the multiplication.

$$\exp\left(\int^t p(s)\,ds\right) y' + p(t)\exp\left(\int^t p(s)\,ds\right) y = q(t)\exp\left(\int^t p(s)\,ds\right)$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\int^t p(s)\,ds\right) y\right] = q(t)\exp\left(\int^t p(s)\,ds\right)$$

Integrate both sides with respect to t.

$$\exp\left(\int^t p(s)\,ds\right) y = \int^t q(r)\exp\left(\int^r p(s)\,ds\right) dr + C_1$$

Divide both sides by $e^{\int^t p(s)\,ds}$ to solve for y.

$$y(t) = \exp\left(-\int^t p(s)\,ds\right)\int^t q(r)\exp\left(\int^r p(s)\,ds\right) dr + C_1\exp\left(-\int^t p(s)\,ds\right)$$

$n = 1$

If $n = 1$, then Bernoulli's equation reduces to

$$y' + p(t)y = q(t)y,$$

which can also be solved by multiplying both sides by an integrating factor I. First, bring $q(t)y$ to the left side and factor y.

$$y' + [p(t) - q(t)]y = 0$$

Use the following integrating factor.

$$I = \exp\left(\int^t [p(r) - q(r)]\,dr\right)$$

Proceed with the multiplication.

$$\exp\left(\int^t [p(r) - q(r)]\,dr\right) y' + [p(t) - q(t)] \exp\left(\int^t [p(r) - q(r)]\,dr\right) y = 0$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\int^t [p(r) - q(r)]\,dr\right) y\right] = 0$$

Integrate both sides with respect to t.

$$\exp\left(\int^t [p(r) - q(r)]\,dr\right) y = C_2$$

Divide both sides by the exponential function to solve for y.

$$y(t) = C_2 \exp\left(-\int^t [p(r) - q(r)]\,dr\right)$$

$n \neq 0, 1$

$$y' + p(t)y = q(t)y^n$$

Divide both sides by y^n.

$$y^{-n}y' + p(t)y^{1-n} = q(t) \tag{1}$$

Make the substitution $u = y^{1-n}$. We now have to find what y' is in terms of this new variable. Differentiate both sides of the substitution with respect to t, using the chain rule on the right side.

$$\frac{du}{dt} = (1 - n)y^{-n} \cdot \frac{dy}{dt}$$

Divide both sides by $1 - n$.

$$\frac{1}{1-n}\frac{du}{dt} = y^{-n}y'$$

Substitute this result and $u = y^{1-n}$ into equation (1) to obtain a linear ODE for u.

$$\frac{1}{1-n}\frac{du}{dt} + p(t)u = q(t)$$