Problem 28

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

$$t^2y' + 2ty - y^3 = 0, \qquad t > 0$$

Solution

Bring y^3 to the right side

$$t^2y' + 2ty = y^3$$

and then divide both sides by t^2 to put the ODE in the proper form.

$$y' + \frac{2}{t}y = \frac{y^3}{t^2}$$

Divide both sides by y^3.

$$y^{-3}y' + \frac{2}{t}y^{-2} = \frac{1}{t^2} \tag{1}$$

Make the substitution $u = y^{-2}$ and differentiate both sides of it with respect to t to find y' in terms of this new variable.

$$\frac{du}{dt} = -2y^{-3} \cdot \frac{dy}{dt}$$

Divide both sides by -2.

$$-\frac{1}{2}\frac{du}{dt} = y^{-3}y'$$

Substitute this result and $u = y^{-2}$ into equation (1) to get a linear ODE for u.

$$-\frac{1}{2}\frac{du}{dt} + \frac{2}{t}u = \frac{1}{t^2}$$

Multiply both sides by -2.

$$\frac{du}{dt} - \frac{4}{t}u = -\frac{2}{t^2}$$

Solve this ODE by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t -\frac{4}{s}\,ds\right) = e^{-4\ln t} = e^{\ln t^{-4}} = t^{-4}$$

Proceed with the multiplication.

$$t^{-4}\frac{du}{dt} - 4t^{-5}u = -2t^{-6}$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}(t^{-4}u) = -2t^{-6}$$

Integrate both sides with respect to t.

$$t^{-4}u = \frac{2}{5}t^{-5} + C$$

Multiply both sides by t^4.

$$u(t) = \frac{2}{5}t^{-1} + Ct^4$$

Now that u is solved for, replace it with y^{-2} and solve for y.

$$y^{-2} = \frac{2}{5t} + Ct^4$$

$$\frac{1}{y^2} = \frac{2 + 5Ct^5}{5t}$$

$$y^2 = \frac{5t}{2 + 5Ct^5}$$

Therefore,

$$y(t) = \pm\sqrt{\frac{5t}{2 + 5Ct^5}}.$$