Problem 28

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

\[ t^2 y' + 2ty - y^3 = 0, \quad t > 0 \]

Solution

Bring \( y^3 \) to the right side

\[ t^2 y' + 2ty = y^3 \]

and then divide both sides by \( t^2 \) to put the ODE in the proper form.

\[ y' + \frac{2}{t} y = \frac{y^3}{t^2} \]

Divide both sides by \( y^3 \).

\[ y^{-3} y' + \frac{2}{t} y^{-2} = \frac{1}{t^2} \quad (1) \]

Make the substitution \( u = y^{-2} \) and differentiate both sides of it with respect to \( t \) to find \( y' \) in terms of this new variable.

\[ \frac{d}{dt} u = -2y^{-3} \cdot \frac{dy}{dt} \]

Divide both sides by \(-2\).

\[ -\frac{1}{2} \frac{du}{dt} = y^{-3}y' \]

Substitute this result and \( u = y^{-2} \) into equation (1) to get a linear ODE for \( u \).

\[ -\frac{1}{2} \frac{du}{dt} + \frac{2}{t} u = \frac{1}{t^2} \]

Multiply both sides by \(-2\).

\[ \frac{du}{dt} - \frac{4}{t} u = -\frac{2}{t^2} \]

Solve this ODE by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int -\frac{4}{s} \, ds \right) = e^{-4\ln t} = e^{\ln t^{-4}} = t^{-4} \]

Proceed with the multiplication.

\[ t^{-4} \frac{du}{dt} - 4t^{-5} u = -2t^{-6} \]

The left side can be written as \( d/dt(Iu) \) by the product rule.

\[ \frac{d}{dt}(t^{-4}u) = -2t^{-6} \]

Integrate both sides with respect to \( t \).

\[ t^{-4}u = \frac{2}{5} t^{-5} + C \]

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Multiply both sides by \( t^4 \).

\[
u(t) = \frac{2}{5} t^{-1} + Ct^4
\]

Now that \( u \) is solved for, replace it with \( y^{-2} \) and solve for \( y \).

\[
y^{-2} = \frac{2}{5t} + Ct^4
\]

\[
\frac{1}{y^2} = \frac{2 + 5Ct^5}{5t}
\]

\[
y^2 = \frac{5t}{2 + 5Ct^5}
\]

Therefore,

\[
y(t) = \pm \sqrt{\frac{5t}{2 + 5Ct^5}}.
\]