Problem 29

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

\[ y' = ry - ky^2, \quad r > 0 \text{ and } k > 0. \]

This equation is important in population dynamics and is discussed in detail in Section 2.5.

Solution

Bring \( ry \) to the left side.

\[ y' - ry = -ky^2 \]

Divide both sides by \( y^2 \).

\[ y^{-2}y' - ry^{-1} = -k \] (1)

Make the substitution \( u = y^{-1} \). Now differentiate both sides of it with respect to \( t \) to find \( y' \) in terms of this new variable.

\[ \frac{du}{dt} = (-1)y^{-2} \cdot \frac{dy}{dt} \]

Multiply both sides by \(-1\).

\[ -\frac{du}{dt} = y^{-2}y' \]

Substitute this result and \( u = y^{-1} \) into equation (1).

\[ -\frac{du}{dt} - ru = -k \]

Multiply both sides by \(-1\).

\[ \frac{du}{dt} + ru = k \]

This ODE can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int t \, r \, ds \right) = e^{rt} \]

Proceed with the multiplication.

\[ e^{rt}\frac{du}{dt} + re^{rt}u = ke^{rt} \]

The left side can be written as \( d/dt(Iu) \) by the product rule.

\[ \frac{d}{dt}(e^{rt}u) = ke^{rt} \]

Integrate both sides with respect to \( t \).

\[ e^{rt}u = \frac{k}{r}e^{rt} + C \]

Divide both sides by \( e^{rt} \).

\[ u(t) = \frac{k}{r} + Ce^{-rt} \]

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Now that \( u \) is solved for, replace it with \( y^{-1} \) and solve for \( y \).

\[
y^{-1} = \frac{k}{r} + Ce^{-rt}
\]

\[
y(t) = \frac{1}{\frac{k}{r} + Ce^{-rt}}
\]

Therefore,

\[
y(t) = \frac{r}{k + Cre^{-rt}}.
\]