Problem 29

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

$$y' = ry - ky^2, \qquad r > 0 \text{ and } k > 0.$$

This equation is important in population dynamics and is discussed in detail in Section 2.5.

Solution

Bring ry to the left side.

$$y' - ry = -ky^2$$

Divide both sides by y^2.

$$y^{-2}y' - ry^{-1} = -k \tag{1}$$

Make the substitution $u = y^{-1}$. Now differentiate both sides of it with respect to t to find y' in terms of this new variable.

$$\frac{du}{dt} = (-1)y^{-2} \cdot \frac{dy}{dt}$$

Multiply both sides by -1.

$$-\frac{du}{dt} = y^{-2}y'$$

Substitute this result and $u = y^{-1}$ into equation (1).

$$-\frac{du}{dt} - ru = -k$$

Multiply both sides by -1.

$$\frac{du}{dt} + ru = k$$

This ODE can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t r\,ds\right) = e^{rt}$$

Proceed with the multiplication.

$$e^{rt}\frac{du}{dt} + re^{rt}u = ke^{rt}$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}(e^{rt}u) = ke^{rt}$$

Integrate both sides with respect to t.

$$e^{rt}u = \frac{k}{r}e^{rt} + C$$

Divide both sides by e^{rt}.

$$u(t) = \frac{k}{r} + Ce^{-rt}$$

Now that u is solved for, replace it with y^{-1} and solve for y.

$$y^{-1} = \frac{k}{r} + Ce^{-rt}$$

$$y(t) = \frac{1}{\frac{k}{r} + Ce^{-rt}}$$

Therefore,

$$y(t) = \frac{r}{k + Cre^{-rt}}.$$