Problem 31

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

$$dy/dt = (\Gamma \cos t + T)y - y^3,$$

where Γ and T are constants. This equation also occurs in the study of the stability of fluid flow.

Solution

Bring the term with y to the left side.

$$y' - (\Gamma \cos t + T)y = -y^3$$

Divide both sides by y^3.

$$y^{-3}y' - (\Gamma \cos t + T)y^{-2} = -1 \tag{1}$$

Make the substitution $u = y^{-2}$ and differentiate both sides of it with respect to t to find what y' is in terms of this new variable.

$$\frac{du}{dt} = (-2)y^{-3} \cdot \frac{dy}{dt}$$

Divide both sides by -2.

$$-\frac{1}{2}\frac{du}{dt} = y^{-3}y'$$

Substitute this result along with $u = y^{-2}$ into equation (1).

$$-\frac{1}{2}\frac{du}{dt} - (\Gamma \cos t + T)u = -1$$

Multiply both sides by -2.

$$\frac{du}{dt} + 2(\Gamma \cos t + T)u = 2$$

This ODE can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t 2(\Gamma \cos s + T)\, ds\right) = e^{2(\Gamma \sin t + Tt)}$$

Proceed with the multiplication.

$$e^{2(\Gamma \sin t + Tt)}\frac{du}{dt} + 2(\Gamma \cos t + T)e^{2(\Gamma \sin t + Tt)}u = 2e^{2(\Gamma \sin t + Tt)}$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}\left[e^{2(\Gamma \sin t + Tt)}u\right] = 2e^{2(\Gamma \sin t + Tt)}$$

Integrate both sides with respect to t.

$$e^{2(\Gamma \sin t + Tt)}u = \int^t 2e^{2(\Gamma \sin s + Ts)}\, ds + C$$

Divide both sides by the exponential function to solve for u.

$$u(t) = \frac{\displaystyle\int^t 2e^{2(\Gamma \sin s + Ts)}\, ds + C}{e^{2(\Gamma \sin t + Tt)}}$$

Now that u is solved for, replace it with y^{-2} and solve for y.

$$y^{-2} = \frac{\displaystyle\int^t 2e^{2(\Gamma \sin s + Ts)}\, ds + C}{e^{2(\Gamma \sin t + Tt)}}$$

$$y^2 = \frac{e^{2(\Gamma \sin t + Tt)}}{\displaystyle\int^t 2e^{2(\Gamma \sin s + Ts)}\, ds + C}$$

Therefore,

$$y(t) = \pm \frac{e^{\Gamma \sin t + Tt}}{\sqrt{\displaystyle\int^t 2e^{2(\Gamma \sin s + Ts)}\, ds + C}}.$$

Note that the lower limit of integration is arbitrary; C will be adjusted to account for whatever choice we make.