Problem 31

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

\[
dy/dt = (\Gamma \cos t + T)y - y^3,
\]

where \( \Gamma \) and \( T \) are constants. This equation also occurs in the study of the stability of fluid flow.

Solution

Bring the term with \( y \) to the left side.

\[
y' - (\Gamma \cos t + T)y = -y^3
\]

Divide both sides by \( y^3 \).

\[
y^{-3}y' - (\Gamma \cos t + T)y^{-2} = -1
\]

(1)

Make the substitution \( u = y^{-2} \) and differentiate both sides of it with respect to \( t \) to find what \( y' \) is in terms of this new variable.

\[
\frac{du}{dt} = (-2)y^{-3} \cdot \frac{dy}{dt}
\]

Divide both sides by \(-2\).

\[
-\frac{1}{2} \frac{du}{dt} = y^{-3}y'
\]

Substitute this result along with \( u = y^{-2} \) into equation (1).

\[
-\frac{1}{2} \frac{du}{dt} - (\Gamma \cos t + T)u = -1
\]

Multiply both sides by \(-2\).

\[
\frac{du}{dt} + 2(\Gamma \cos t + T)u = 2
\]

This ODE can be solved by multiplying both sides by an integrating factor \( I \).

\[
I = \exp \left(\int 2(\Gamma \cos s + T) \, ds\right) = e^{2(\Gamma \sin t + Tt)}
\]

Proceed with the multiplication.

\[
e^{2(\Gamma \sin t + Tt)} \frac{du}{dt} + 2(\Gamma \cos t + T)e^{2(\Gamma \sin t + Tt)}u = 2e^{2(\Gamma \sin t + Tt)}
\]

The left side can be written as \( d/dt(\text{Iu}) \) by the product rule.

\[
\frac{d}{dt} \left[ e^{2(\Gamma \sin t + Tt)}u \right] = 2e^{2(\Gamma \sin t + Tt)}
\]

Integrate both sides with respect to \( t \).

\[
e^{2(\Gamma \sin t + Tt)}u = \int^t e^{2(\Gamma \sin s + Ts)} \, ds + C
\]

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Divide both sides by the exponential function to solve for \( u \).

\[
u(t) = \int_t^t 2e^{2(\Gamma \sin s + Ts)} \, ds + C \]

Now that \( u \) is solved for, replace it with \( y^{-2} \) and solve for \( y \).

\[
y^{-2} = \int_t^t 2e^{2(\Gamma \sin s + Ts)} \, ds + C
\]

\[
y^2 = \int_t^t 2e^{2(\Gamma \sin s + Ts)} \, ds + C
\]

Therefore,

\[
y(t) = \pm \frac{e^{\Gamma \sin t + Tt}}{\sqrt{\int_t^t 2e^{2(\Gamma \sin s + Ts)} \, ds + C}}.
\]

Note that the lower limit of integration is arbitrary; \( C \) will be adjusted to account for whatever choice we make.