Problem 15

Suppose that a certain population obeys the logistic equation $dy/dt = ry[1-(y/K)]$.

(a) If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

(b) If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha,\ \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.

Solution

The initial value problem to solve here is

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right), \quad y(0) = y_0.$$

Solve this ODE by separating variables.

$$\frac{dy}{y\left(1 - \frac{y}{K}\right)} = r\,dt$$

Integrate both sides, using partial fraction decomposition to evaluate the integral on the left side.

$$\int^y \frac{ds}{s\left(1 - \frac{s}{K}\right)} = rt + C$$

$$\int^y \left(\frac{1}{s} + \frac{1/K}{1 - \frac{s}{K}}\right) ds = rt + C$$

$$\int^y \frac{ds}{s} - \int^y \frac{ds}{s-K} = rt + C$$

$$\ln|s|\Big|^y - \ln|s-K|\Big|^y = rt + C$$

$$\ln|y| - \ln|y-K| = rt + C$$

$$\ln\left|\frac{y}{y-K}\right| = rt + C$$

Exponentiate both sides.

$$\begin{aligned}\left|\frac{y}{y-K}\right| &= e^{rt+C} \\ &= e^C e^{rt}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\frac{y}{y-K} = \pm e^C e^{rt}$$

Use a new constant A for $\pm e^C$.

$$\frac{y}{y-K} = Ae^{rt}$$

Multiply both sides by $y - K$.

$$y = Aye^{rt} - AKe^{rt}$$
$$y(1 - Ae^{rt}) = -AKe^{rt}$$

Consequently, the general solution is

$$\begin{aligned} y(t) &= \frac{-AKe^{rt}}{1 - Ae^{rt}} \\ &= \frac{AKe^{rt}}{Ae^{rt} - 1}. \end{aligned}$$

Apply the initial condition $y(0) = y_0$ now to determine A: set $y(t) = y_0$ and $t = 0$ and then solve the resulting equation for A.

$$y_0 = \frac{AK}{A - 1} \quad \rightarrow \quad A = \frac{y_0}{y_0 - K}$$

Then the general solution becomes

$$\begin{aligned} y(t) &= \frac{\frac{y_0}{y_0 - K} Ke^{rt}}{\frac{y_0}{y_0 - K} e^{rt} - 1} \\ &= \frac{Ky_0e^{rt}}{y_0e^{rt} - (y_0 - K)}. \end{aligned}$$

Therefore,

$$y(t) = \frac{Ky_0e^{rt}}{y_0(e^{rt} - 1) + K}.$$

Part (a)

To find the time τ at which the population has doubled to $2K/3$, set $y(t) = 2K/3$, $y_0 = K/3$, and $t = \tau$ and then solve the resulting equation for τ.

$$\frac{2K}{3} = \frac{K\frac{K}{3}e^{r\tau}}{\frac{K}{3}(e^{r\tau} - 1) + K}$$
$$\frac{2K}{3} = \frac{\frac{K}{3}e^{r\tau}}{\frac{1}{3}(e^{r\tau} - 1) + 1}$$
$$2 = \frac{e^{r\tau}}{\frac{1}{3}(e^{r\tau} - 1) + 1}$$
$$2 = \frac{3e^{r\tau}}{e^{r\tau} - 1 + 3}$$
$$2e^{r\tau} + 4 = 3e^{r\tau}$$
$$e^{r\tau} = 4$$
$$r\tau = \ln 4$$

Therefore,

$$\tau = \frac{\ln 4}{r}.$$

If $r = 0.025$ year^{-1}, then

$$\tau = \frac{\ln 4}{0.025} \text{ years} \approx 55.5 \text{ years.}$$

Part (b)

Set $y_0 = \alpha K$, $y(t) = \beta K$, and $t = T$ and then solve the resulting equation for T.

$$\beta K = \frac{K(\alpha K)e^{rT}}{\alpha K(e^{rT} - 1) + K}$$

$$\beta K = \frac{(\alpha K)e^{rT}}{\alpha(e^{rT} - 1) + 1}$$

$$\beta = \frac{\alpha e^{rT}}{\alpha(e^{rT} - 1) + 1}$$

$$\alpha\beta(e^{rT} - 1) + \beta = \alpha e^{rT}$$

$$\alpha\beta e^{rT} - \alpha\beta + \beta = \alpha e^{rT}$$

$$(\alpha\beta - \alpha)e^{rT} - \alpha\beta + \beta = 0$$

$$\alpha(\beta - 1)e^{rT} - \beta(\alpha - 1) = 0$$

$$e^{rT} = \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

$$rT = \ln \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

$$T = \frac{1}{r} \ln \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

Therefore,

$$T = \frac{1}{r} \ln \frac{\beta(1 - \alpha)}{\alpha(1 - \beta)}.$$

If $r = 0.025$ year^{-1}, $\alpha = 0.1$, and $\beta = 0.9$, then

$$T = \frac{1}{0.025} \ln \frac{0.9(1 - 0.1)}{0.1(1 - 0.9)} \text{ years} \approx 176 \text{ years}.$$