Problem 15

Suppose that a certain population obeys the logistic equation \( \frac{dy}{dt} = ry\left[1 - \frac{y}{K}\right] \).

(a) If \( y_0 = K/3 \), find the time \( \tau \) at which the initial population has doubled. Find the value of \( \tau \) corresponding to \( r = 0.025 \) per year.

(b) If \( y_0/K = \alpha \), find the time \( T \) at which \( y(T)/K = \beta \), where \( 0 < \alpha, \beta < 1 \). Observe that \( T \to \infty \) as \( \alpha \to 0 \) or as \( \beta \to 1 \). Find the value of \( T \) for \( r = 0.025 \) per year, \( \alpha = 0.1 \), and \( \beta = 0.9 \).

Solution

The initial value problem to solve here is

\[
\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right), \quad y(0) = y_0.
\]

Solve this ODE by separating variables.

\[
\frac{dy}{y \left(1 - \frac{y}{K}\right)} = r \, dt
\]

Integrate both sides, using partial fraction decomposition to evaluate the integral on the left side.

\[
\int_{y_0}^{y} \frac{ds}{s \left(1 - \frac{s}{K}\right)} = rt + C
\]

Use the partial fraction decomposition

\[
\frac{1}{s \left(1 - \frac{s}{K}\right)} = \frac{1}{s} + \frac{1/K}{1 - \frac{s}{K}}
\]

\[
\int_{y_0}^{y} \left(\frac{1}{s} + \frac{1/K}{1 - \frac{s}{K}}\right) ds = rt + C
\]

\[
\int_{y_0}^{y} \frac{ds}{s} - \int_{y_0}^{y} \frac{ds}{s - K} = rt + C
\]

\[
\ln |s| \bigg|_{y_0}^{y} - \ln |s - K| \bigg|_{y_0}^{y} = rt + C
\]

\[
\ln \left|\frac{y}{y - K}\right| = rt + C
\]

Exponentiate both sides.

\[
\left|\frac{y}{y - K}\right| = e^{rt+C}
\]

\[
\left|\frac{y}{y - K}\right| = e^{C} e^{rt}
\]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[
\frac{y}{y - K} = \pm e^{C} e^{rt}
\]

Use a new constant \( A \) for \( \pm e^{C} \).

\[
\frac{y}{y - K} = Ae^{rt}
\]

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Multiply both sides by $y - K$.

\[ y = Ay e^{rt} - AK e^{rt} \]

\[ y(1 - Ae^{rt}) = -AK e^{rt} \]

Consequently, the general solution is

\[ y(t) = \frac{-AK e^{rt}}{1 - Ae^{rt}} = \frac{AK e^{rt}}{A e^{rt} - 1}. \]

Apply the initial condition $y(0) = y_0$ now to determine $A$: set $y(t) = y_0$ and $t = 0$ and then solve the resulting equation for $A$.

\[ y_0 = \frac{AK}{A - 1} \rightarrow A = \frac{y_0}{y_0 - K} \]

Then the general solution becomes

\[ y(t) = \frac{y_0}{y_0 - K} e^{rt} \frac{y_0 e^{rt}}{y_0 - K e^{rt} - (y_0 - K)}. \]

Therefore,

\[ y(t) = \frac{K y_0 e^{rt}}{y_0 (e^{rt} - 1) + K}. \]

**Part (a)**

To find the time $\tau$ at which the population has doubled to $2K/3$, set $y(t) = 2K/3$, $y_0 = K/3$, and $t = \tau$ and then solve the resulting equation for $\tau$.

\[ \frac{2K}{3} = \frac{K}{3} e^{r \tau} \frac{K e^{r \tau}}{(e^{r \tau} - 1) + K} \]

\[ \frac{2K}{3} = \frac{1}{3} e^{r \tau} + 1 \]

\[ 2 = \frac{3}{3} e^{r \tau} + 1 \]

\[ 2e^{r \tau} + 4 = 3e^{r \tau} \]

\[ e^{r \tau} = 4 \]

\[ r \tau = \ln 4 \]

Therefore,

\[ \tau = \frac{\ln 4}{r}. \]

If $r = 0.025 \text{ year}^{-1}$, then

\[ \tau = \frac{\ln 4}{0.025} \text{ years} \approx 55.5 \text{ years}. \]
Part (b)

Set $y_0 = \alpha K$, $y(t) = \beta K$, and $t = T$ and then solve the resulting equation for $T$.

$$\beta K = \frac{K(\alpha K)e^{rT}}{\alpha K(e^{rT} - 1) + K}$$

$$\beta K = \frac{(\alpha K)e^{rT}}{\alpha(e^{rT} - 1) + 1}$$

$$\beta = \frac{\alpha e^{rT}}{\alpha(e^{rT} - 1) + 1}$$

$$\alpha \beta (e^{rT} - 1) + \beta = \alpha e^{rT}$$

$$\alpha \beta e^{rT} - \alpha \beta + \beta = \alpha e^{rT}$$

$$(\alpha \beta - \alpha)e^{rT} - \alpha \beta + \beta = 0$$

$$\alpha(\beta - 1)e^{rT} - \beta(\alpha - 1) = 0$$

$$e^{rT} = \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

$$rT = \ln \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

$$T = \frac{1}{r} \ln \frac{\beta(\alpha - 1)}{\alpha(\beta - 1)}$$

Therefore,

$$T = \frac{1}{r} \ln \frac{\beta(1 - \alpha)}{\alpha(1 - \beta)}$$

If $r = 0.025$ year$^{-1}$, $\alpha = 0.1$, and $\beta = 0.9$, then

$$T = \frac{1}{0.025} \ln \frac{0.9(1 - 0.1)}{0.1(1 - 0.9)} \text{ years} \approx 176 \text{ years}.$$