# Problem 18

A pond forms as water collects in a conical depression of radius $a$ and depth $h$. Suppose that water flows in at a constant rate $k$ and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time $t$ satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$

where $\alpha$ is the coefficient of evaporation.

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

(c) Find a condition that must be satisfied if the pond is not to overflow.

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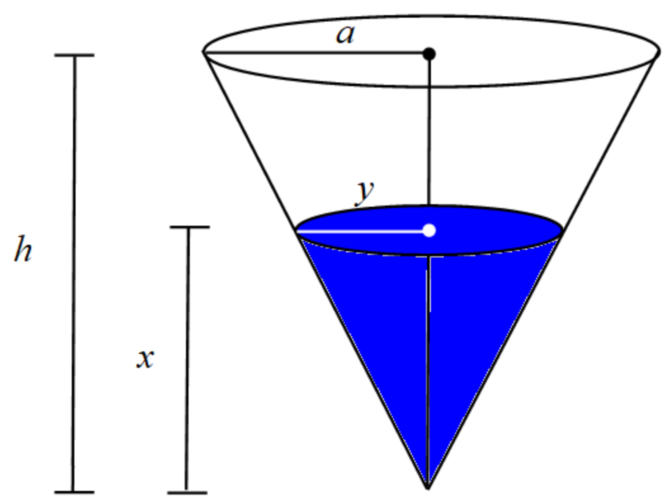
**Solution**

## Part (a)

Start by drawing a schematic of the pond.




According to the law of conservation of mass, matter is neither created nor destroyed. If water flows into the pond at some rate, then it must leave at that same rate or else it will accumulate.

$$\text{rate of accumulation} = \text{rate in} - \text{rate out}$$

If $m$ denotes the mass of the water, then $dm/dt$ represents the rate that it accumulates. $k$ is the volume of water per unit time that flows into the pond, so multiply it by the density $\rho$ to get a mass flow rate. The rate that volume is lost by evaporation is $\alpha S$, where $\alpha$ is a proportionality constant and $S$ is the surface area of the water exposed to the air. Multiply this result by $\rho$ to get a mass flow rate.

$$\frac{dm}{dt} = \rho k - \rho\alpha S$$

Mass is density times volume.

$$\frac{d(\rho V)}{dt} = \rho k - \rho\alpha S$$

$$\rho\frac{dV}{dt} = \rho k - \rho\alpha S$$

Divide both sides by $\rho$.

$$\frac{dV}{dt} = k - \alpha S$$

Our task now is to write the surface area in terms of the volume. As shown in the schematic, the exposed area is a circle, so $S = \pi y^2$. Because the triangles are similar, the distances are related by

$$\frac{y}{x} = \frac{a}{h} \quad \rightarrow \quad y = \frac{a}{h}x.$$

Substitute this expression for $y$ into the previous equation.

$$\begin{aligned}\frac{dV}{dt} &= k - \alpha\pi y^2 \\ &= k - \alpha\pi\left(\frac{a}{h}x\right)^2 \\ &= k - \alpha\pi\frac{a^2}{h^2}x^2 \qquad (1)\end{aligned}$$

The volume of the pond is obtained by integrating the cross-sectional area over a distance $x$.

$$V = \int_0^x \pi y^2\,dx = \int_0^x \pi\left(\frac{a}{h}x\right)^2 dx = \frac{\pi a^2}{h^2}\int_0^x x^2\,dx = \frac{\pi a^2}{3h^2}x^3 \qquad (2)$$

Solve this equation for $x$

$$\left(\frac{3h^2}{\pi a^2}V\right)^{1/3} = x$$

and plug the result into equation (1).

$$\begin{aligned}\frac{dV}{dt} &= k - \alpha\pi\frac{a^2}{h^2}\left(\frac{3h^2}{\pi a^2}V\right)^{2/3} \\ &= k - \alpha\pi\left(\frac{a^3}{h^3}\cdot\frac{3h^2}{\pi a^2}V\right)^{2/3} \\ &= k - \alpha\pi\left(\frac{3a}{\pi h}V\right)^{2/3}\end{aligned}$$

Therefore, the governing equation for the volume in the pond is

$$\frac{dV}{dt} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3}V^{2/3}.$$

## Part (b)

If equilibrium occurs, then there will be no accumulation. Set $dV/dt = 0$ and solve for $V$.

$$0 = k - \alpha\pi \left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}$$

$$\alpha\pi \left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3} = k$$

$$V^{2/3} = \frac{k}{\alpha\pi} \left(\frac{\pi h}{3a}\right)^{2/3}$$

$$V = \left(\frac{k}{\alpha\pi}\right)^{3/2} \frac{\pi h}{3a}$$

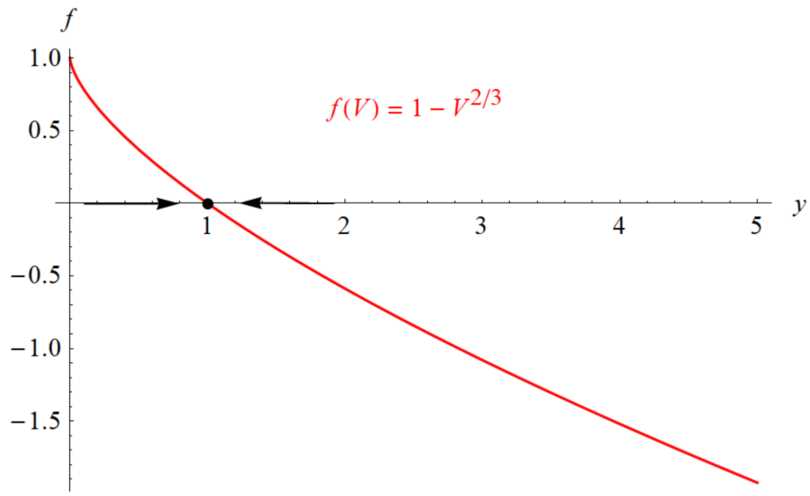
Now substitute equation (2) for $V$ and solve for $x$, the height.

$$\frac{\pi a^2}{3h^2} x^3 = \left(\frac{k}{\alpha\pi}\right)^{3/2} \frac{\pi h}{3a}$$

$$x^3 = \frac{h^3}{a^3} \left(\frac{k}{\alpha\pi}\right)^{3/2}$$

Therefore, the height at equilibrium is

$$x = \frac{h}{a}\sqrt{\frac{k}{\alpha\pi}}.$$




Plotting $f(V) = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$ versus $V$ with the coefficients set to 1, we see that this equilibrium is stable.

## Part (c)

Return to the ODE for $V$.

$$\frac{dV}{dt} = k - \alpha\pi \left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}$$

In order for the pond to not overflow, the rate of accumulation cannot be positive when $x = h$. The volume at this height, according to equation (2), is $V = \pi a^2 h/3$.

$$\frac{dV}{dt} = k - \alpha\pi \left(\frac{3a}{\pi h}\right)^{2/3} \left(\frac{\pi a^2 h}{3}\right)^{2/3} \leq 0$$

$$k - \alpha\pi \left(\frac{3a}{\pi h} \cdot \frac{\pi a^2 h}{3}\right)^{2/3} \leq 0$$

$$k - \alpha\pi (a^3)^{2/3} \leq 0$$

$$k - \alpha\pi a^2 \leq 0$$

$$k \leq \alpha\pi a^2$$

Therefore, the condition for overflow to be prevented is

$$\frac{k}{\alpha} \leq \pi a^2.$$