Problem 3
Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

\[(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0\]

Solution
The ODE is exact because

\[\frac{\partial}{\partial y}(3x^2 - 2xy + 2) = \frac{\partial}{\partial x}(6y^2 - x^2 + 3) = -2x.\]

That means there exists a potential function \(\psi = \psi(x, y)\) such that

\[\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2\]  \(\quad (1)\)

\[\frac{\partial \psi}{\partial y} = 6y^2 - x^2 + 3.\]  \(\quad (2)\)

Integrate both sides of equation (1) partially with respect to \(x\) to get \(\psi\).

\[\psi(x, y) = x^3 - x^2y + 2x + f(y)\]

Here \(f\) is an arbitrary function of \(y\). Differentiate both sides with respect to \(y\).

\[\psi_y(x, y) = -x^2 + f'(y)\]

Comparing this to equation (2), we see that

\[f'(y) = 6y^2 + 3 \quad \rightarrow \quad f(y) = 2y^3 + 3y.\]

As a result, a potential function is

\[\psi(x, y) = x^3 - x^2y + 2x + 2y^3 + 3y.\]

Notice that by substituting equations (1) and (2), the ODE can be written as

\[\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.\]  \(\quad (3)\)

Recall that the differential of \(\psi(x, y)\) is defined as

\[d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.\]

Dividing both sides by \(dx\), we obtain the fundamental relationship between the total derivative of \(\psi\) and its partial derivatives.

\[\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}\]

With it, equation (3) becomes

\[\frac{d\psi}{dx} = 0.\]

Integrate both sides with respect to \(x\).

\[\psi(x, y) = C\]

Therefore,

\[x^3 - x^2y + 2x + 2y^3 + 3y = C.\]
This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.