Problem 5

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

\[
\frac{dy}{dx} = -\frac{ax + by}{bx + cy}
\]

Solution

Multiply both sides by \(bx + cy\).

\[(bx + cy)\frac{dy}{dx} = -(ax + by)\]

Add \(ax + by\) to both sides.

\[(ax + by) + (bx + cy)\frac{dy}{dx} = 0\]

The ODE is exact because

\[
\frac{\partial}{\partial y}(ax + by) = \frac{\partial}{\partial x}(bx + cy) = b.
\]

That means there exists a potential function \(\psi = \psi(x, y)\) such that

\[
\frac{\partial \psi}{\partial x} = ax + by \quad (1)
\]

\[
\frac{\partial \psi}{\partial y} = bx + cy. \quad (2)
\]

Integrate both sides of equation (1) partially with respect to \(x\) to get \(\psi\).

\[
\psi(x, y) = \frac{ax^2}{2} + bxy + f(y)
\]

Here \(f\) is an arbitrary function of \(y\). Differentiate both sides with respect to \(y\).

\[
\psi_y(x, y) = bx + f'(y)
\]

Comparing this to equation (2), we see that

\[
f'(y) = cy \quad \rightarrow \quad f(y) = \frac{cy^2}{2}.
\]

As a result, a potential function is

\[
\psi(x, y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2}.
\]

Notice that by substituting equations (1) and (2), the ODE can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (3)
\]

Recall that the differential of \(\psi(x, y)\) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

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Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}
\]

With it, equation (3) becomes

\[
\frac{d\psi}{dx} = 0.
\]

Integrate both sides with respect to \( x \).

\[
\psi(x, y) = C
\]

Therefore,

\[
\frac{ax^2}{2} + bxy + \frac{cy^2}{2} = C.
\]