Problem 5

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$$

Solution

Multiply both sides by $bx + cy$.

$$(bx+cy)\frac{dy}{dx} = -(ax+by)$$

Add $ax + by$ to both sides.

$$(ax+by) + (bx+cy)\frac{dy}{dx} = 0$$

The ODE is exact because

$$\frac{\partial}{\partial y}(ax+by) = \frac{\partial}{\partial x}(bx+cy) = b.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = ax + by \quad (1)$$

$$\frac{\partial \psi}{\partial y} = bx + cy. \quad (2)$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x,y) = \frac{ax^2}{2} + bxy + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y) = bx + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = cy \quad \to \quad f(y) = \frac{cy^2}{2}.$$

As a result, a potential function is

$$\psi(x,y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2}.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \quad (3)$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$\frac{ax^2}{2} + bxy + \frac{cy^2}{2} = C.$$