

Problem 7

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)y' = 0$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(e^x \sin y - 2y \sin x) = \frac{\partial}{\partial x}(e^x \cos y + 2 \cos x) = e^x \cos y - 2 \sin x.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x \tag{1}$$

$$\frac{\partial \psi}{\partial y} = e^x \cos y + 2 \cos x. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ .

$$\psi(x, y) = e^x \sin y + 2y \cos x + f(y)$$

Here f is an arbitrary function of y . Differentiate both sides with respect to y .

$$\psi_y(x, y) = e^x \cos y + 2 \cos x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 0 \quad \rightarrow \quad f(y) = 0.$$

As a result, a potential function is

$$\psi(x, y) = e^x \sin y + 2y \cos x.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

Dividing both sides by dx , we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

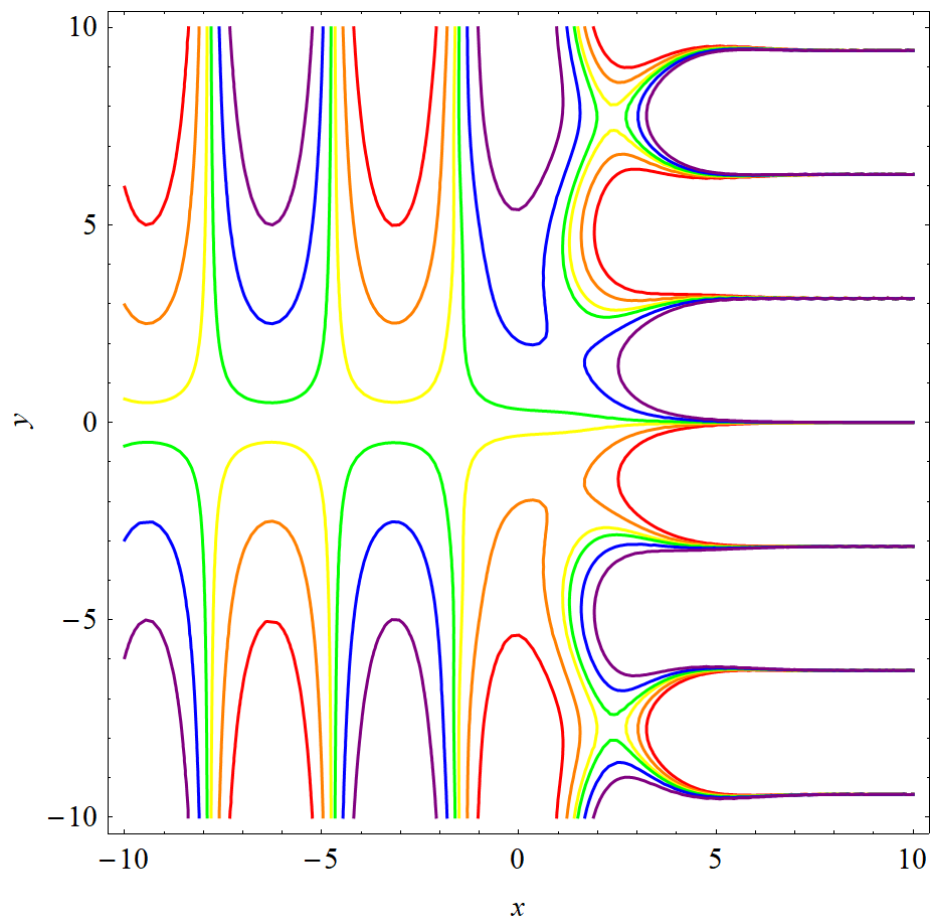
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x .

$$\psi(x, y) = C$$

Therefore,

$$e^x \sin y + 2y \cos x = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.

Notice also that $y = 0$ satisfies the ODE.