

## Problem 9

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x) + (xe^{xy} \cos 2x - 3)y' = 0$$

### Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x) = \frac{\partial}{\partial x}(xe^{xy} \cos 2x - 3) = e^{xy}[(1 + xy) \cos 2x - 2x \sin 2x].$$

That means there exists a potential function  $\psi = \psi(x, y)$  such that

$$\frac{\partial \psi}{\partial x} = ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x \tag{1}$$

$$\frac{\partial \psi}{\partial y} = xe^{xy} \cos 2x - 3. \tag{2}$$

Integrate both sides of equation (2) partially with respect to  $y$  to get  $\psi$ .

$$\psi(x, y) = e^{xy} \cos 2x - 3y + f(x)$$

Here  $f$  is an arbitrary function of  $x$ . Differentiate both sides with respect to  $x$ .

$$\psi_x(x, y) = ye^{xy} \cos 2x - 2e^{xy} \sin 2x + f'(x)$$

Comparing this to equation (1), we see that

$$f'(x) = 2x \quad \rightarrow \quad f(x) = x^2.$$

As a result, a potential function is

$$\psi(x, y) = e^{xy} \cos 2x - 3y + x^2.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of  $\psi(x, y)$  is defined as

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

Dividing both sides by  $dx$ , we obtain the fundamental relationship between the total derivative of  $\psi$  and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

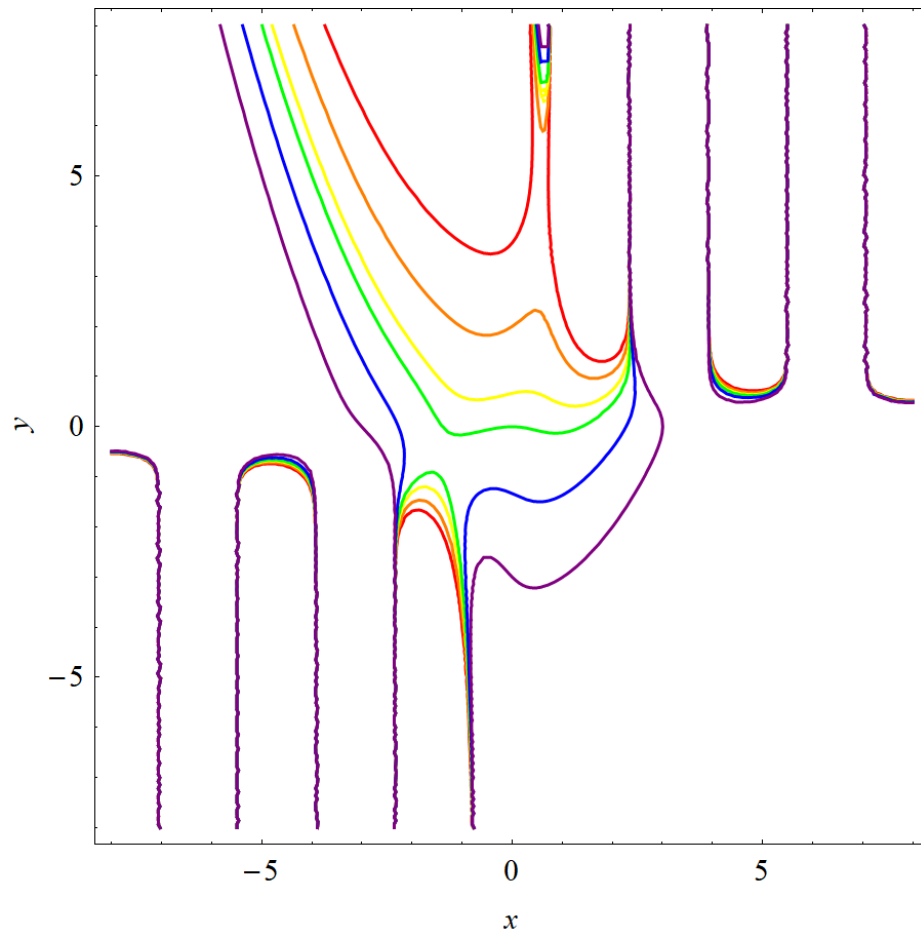
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to  $x$ .

$$\psi(x, y) = C$$

Therefore,

$$e^{xy} \cos 2x - 3y + x^2 = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are  $C = -10$ ,  $C = -5$ ,  $C = -1$ ,  $C = 1$ ,  $C = 5$ , and  $C = 10$ , respectively.