Problem 14

In each of Problems 13 and 14, solve the given initial value problem and determine at least approximately where the solution is valid.

\[(9x^2 + y - 1) - (4y - x)y' = 0, \quad y(1) = 0\]

Solution

Distribute the minus sign in the second term.

\[(9x^2 + y - 1) + (x - 4y)y' = 0\]

The ODE is exact because

\[
\frac{\partial}{\partial y}(9x^2 + y - 1) = \frac{\partial}{\partial x}(x - 4y) = 1.
\]

That means there exists a potential function \(\psi(x, y)\) such that

\[
\frac{\partial \psi}{\partial x} = 9x^2 + y - 1 \quad (1)
\]

\[
\frac{\partial \psi}{\partial y} = x - 4y. \quad (2)
\]

Integrate both sides of equation (1) partially with respect to \(x\) to get \(\psi\).

\[
\psi(x, y) = 3x^3 + xy - x + f(y)
\]

Here \(f\) is an arbitrary function of \(y\). Differentiate both sides with respect to \(y\).

\[
\psi_y(x, y) = x + f'(y)
\]

Comparing this to equation (2), we see that

\[
f'(y) = -4y \quad \rightarrow \quad f(y) = -2y^2.
\]

As a result, a potential function is

\[
\psi(x, y) = 3x^3 + xy - x - 2y^2.
\]

Notice that by substituting equations (1) and (2), the ODE can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (3)
\]

Recall that the differential of \(\psi(x, y)\) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

Dividing both sides by \(dx\), we obtain the fundamental relationship between the total derivative of \(\psi\) and its partial derivatives.

\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}
\]

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With it, equation (3) becomes

\[ \frac{d\psi}{dx} = 0. \]

Integrate both sides with respect to \( x \).

\[ \psi(x, y) = C \]

\[ 3x^3 + xy - x - 2y^2 = C \]

Apply the condition \( y(1) = 0 \) now to determine \( C \).

\[ 3(1)^3 + (1)(0) - (1) - 2(0)^2 = C \quad \rightarrow \quad C = 2 \]

Therefore,

\[ 3x^3 + xy - x - 2y^2 = 2. \]

Solve for \( y \) using the quadratic formula.

\[ -2y^2 + xy - x + 3x^3 - 2 = 0 \]

\[ y = \frac{-x \pm \sqrt{x^2 - 4(-2)(-x + 3x^3 - 2)}}{2(-2)} \]

\[ y = \frac{-x \pm \sqrt{24x^3 + x^2 - 8x - 16}}{-4} \]

Below is a plot of these two functions, and the condition \( y(1) = 0 \) is represented by a green dot.

The blue curve is in contact with the dot, so the solution to the initial value problem is

\[ y = \frac{-x + \sqrt{24x^3 + x^2 - 8x - 16}}{-4}, \]

which is valid roughly when

\[ x \gtrsim 0.98. \]