Problem 15

In each of Problems 15 and 16, find the value of b for which the given equation is exact, and then solve it using that value of b.

$$(xy^2 + bx^2y) + (x + y)x^2y' = 0$$

Solution

The ODE is exact only if

$$\frac{\partial}{\partial y}(xy^2 + bx^2y) = 2xy + bx^2 = \frac{\partial}{\partial x}[(x + y)x^2] = 3x^2 + 2xy,$$

that is, $b = 3$. That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = xy^2 + 3x^2y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = (x + y)x^2. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = x^2y + x^3 + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 0 \quad \rightarrow \quad f(y) = 0.$$

As a result, a potential function is

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

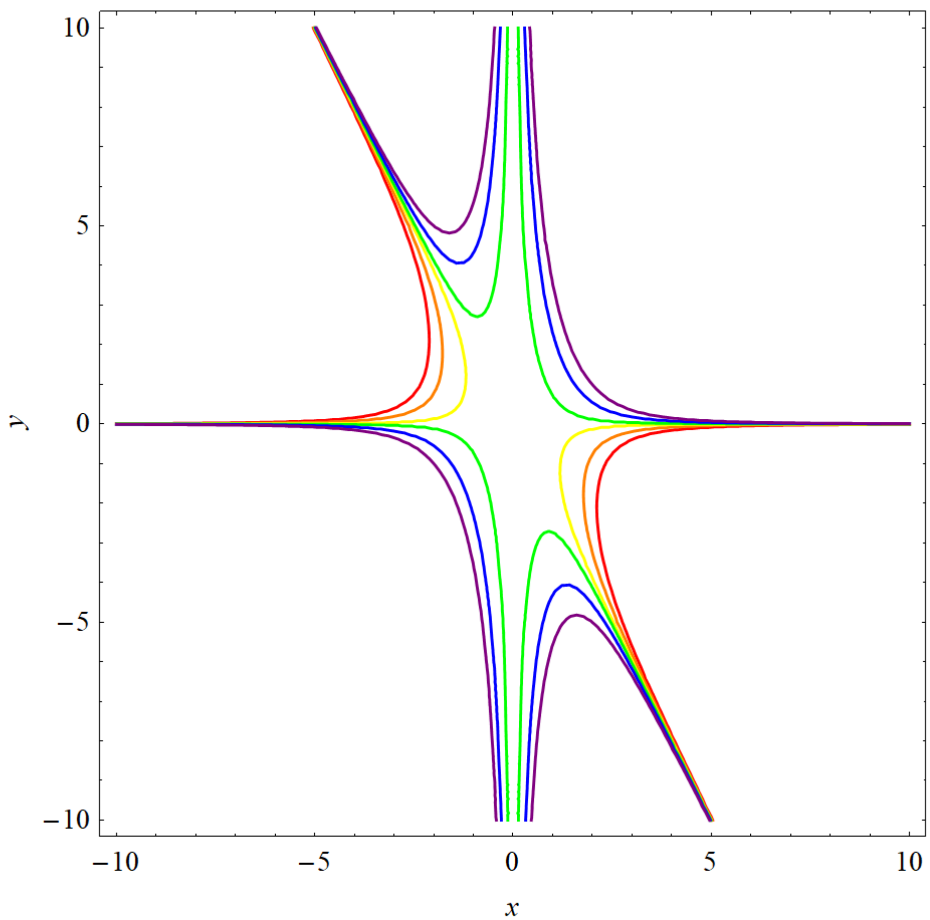
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$\frac{x^2y^2}{2} + x^3y = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.