Problem 17

Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle $R$ and is therefore exact. Show that a possible function $\psi(x,y)$ is

$$
\psi(x,y) = \int_{x_0}^{x} M(s,y_0) \, ds + \int_{y_0}^{y} N(x,t) \, dt,
$$

where $(x_0,y_0)$ is a point in $R$.

Solution

Equation (6) is

$$
M(x,y) + N(x,y)y' = 0.
$$

Assuming it is exact, there exists a potential function $\psi = \psi(x,y)$ such that

$$
\frac{\partial \psi}{\partial x} = M(x,y) \quad (1)
$$

$$
\frac{\partial \psi}{\partial y} = N(x,y). \quad (2)
$$

Integrate both sides of equation (1) partially with respect to $x$ to get $\psi$.

$$
\psi(x,y) = \int_{x_0}^{x} M(s,y) \, ds + f(y)
$$

Here $f(y)$ is an arbitrary function of $y$. As a result, the lower limit of integration is arbitrary as well and can be set to $x_0$.

$$
\psi(x,y) = \int_{x_0}^{x} M(s,y) \, ds + f(y)
$$

Differentiate both sides with respect to $y$.

$$
\psi_y(x,y) = \int_{x_0}^{x} M_y(s,y) \, ds + f'(y)
$$

Comparing this to equation (2), we see that

$$
\int_{x_0}^{x} M_y(s,y) \, ds + f'(y) = N(x,y) \quad \rightarrow \quad f'(y) = N(x,y) - \int_{x_0}^{x} M_y(s,y) \, ds.
$$

Integrate both sides from $y_0$ to $y$ to get $f(y)$.

$$
f(y) = \int_{y_0}^{y} N(x,t) \, dt - \int_{y_0}^{y} \int_{x_0}^{x} M_t(s,t) \, ds \, dt
$$

$$
= \int_{y_0}^{y} N(x,t) \, dt - \int_{x_0}^{x} \int_{y_0}^{y} M_t(s,t) \, dt \, ds
$$

$$
= \int_{y_0}^{y} N(x,t) \, dt - \int_{x_0}^{x} \left[ M(s,y) - M(s,y_0) \right] \, ds
$$

$$
= \int_{y_0}^{y} N(x,t) \, dt - \int_{x_0}^{x} M(s,y) \, ds + \int_{x_0}^{x} M(s,y_0) \, ds
$$

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Consequently, a potential function is

\[
\psi(x, y) = \int_{x_0}^{x} M(s, y) \, ds + \int_{y_0}^{y} N(x, t) \, dt - \int_{x_0}^{x} M(s, y) \, ds + \int_{x_0}^{x} M(s, y_0) \, ds
\]

\[
= \int_{x_0}^{x} M(s, y_0) \, ds + \int_{y_0}^{y} N(x, t) \, dt,
\]

where \((x_0, y_0)\) is a point in \(R\).