Problem 17

Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x,y)$ is

$$\psi(x,y) = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt,$$

where (x_0, y_0) is a point in R.

Solution

Equation (6) is

$$M(x,y) + N(x,y)y' = 0.$$

Assuming it is exact, there exists a potential function $\psi = \psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial \psi}{\partial y} = N(x,y). \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x,y) = \int^{x} M(s,y)\,ds + f(y)$$

Here $f(y)$ is an arbitrary function of y. As a result, the lower limit of integration is arbitrary as well and can be set to x_0.

$$\psi(x,y) = \int_{x_0}^{x} M(s,y)\,ds + f(y)$$

Differentiate both sides with respect to y.

$$\psi_y(x,y) = \int_{x_0}^{x} M_y(s,y)\,ds + f'(y)$$

Comparing this to equation (2), we see that

$$\int_{x_0}^{x} M_y(s,y)\,ds + f'(y) = N(x,y) \quad \rightarrow \quad f'(y) = N(x,y) - \int_{x_0}^{x} M_y(s,y)\,ds.$$

Integrate both sides from y_0 to y to get $f(y)$.

$$\begin{aligned}
f(y) &= \int_{y_0}^{y} N(x,t)\,dt - \int_{y_0}^{y}\int_{x_0}^{x} M_t(s,t)\,ds\,dt \\
&= \int_{y_0}^{y} N(x,t)\,dt - \int_{x_0}^{x}\int_{y_0}^{y} M_t(s,t)\,dt\,ds \\
&= \int_{y_0}^{y} N(x,t)\,dt - \int_{x_0}^{x} [M(s,y) - M(s,y_0)]\,ds \\
&= \int_{y_0}^{y} N(x,t)\,dt - \int_{x_0}^{x} M(s,y)\,ds + \int_{x_0}^{x} M(s,y_0)\,ds
\end{aligned}$$

Consequently, a potential function is

$$\begin{aligned}\psi(x,y) &= \cancel{\int_{x_0}^{x} M(s,y)\,ds} + \int_{y_0}^{y} N(x,t)\,dt - \cancel{\int_{x_0}^{x} M(s,y)\,ds} + \int_{x_0}^{x} M(s,y_0)\,ds \\ &= \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt,\end{aligned}$$

where (x_0, y_0) is a point in R.