Problem 25

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$$

Solution

The ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2 \neq \frac{\partial}{\partial x}(x^2 + y^2) = 2x.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(3x^2y + 2xy + y^3)\mu + (x^2 + y^2)\mu y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(3x^2y + 2xy + y^3)\mu] = \frac{\partial}{\partial x}[(x^2 + y^2)\mu].$$

Expand both sides.

$$(3x^2 + 2x + 3y^2)\mu + (3x^2y + 2xy + y^3)\frac{\partial \mu}{\partial y} = 2x\mu + (x^2 + y^2)\frac{\partial \mu}{\partial x}$$

$$3(x^2 + y^2)\mu + (3x^2y + 2xy + y^3)\frac{\partial \mu}{\partial y} = (x^2 + y^2)\frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$3(x^2 + y^2)\mu = (x^2 + y^2)\frac{d\mu}{dx}$$

Solve for μ.

$$\frac{d\mu}{dx} = 3\mu$$

$$\frac{\frac{d\mu}{dx}}{\mu} = 3$$

$$\frac{d}{dx}\ln \mu = 3$$

$$\ln \mu = 3x$$

As a result, an integrating factor is

$$\mu(x) = e^{3x}.$$

Multiply both sides of the original ODE by e^{3x}.

$$(3x^2e^{3x}y + 2xe^{3x}y + e^{3x}y^3) + (x^2e^{3x} + e^{3x}y^2)y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 3x^2 e^{3x} y + 2x e^{3x} y + e^{3x} y^3 \tag{1}$$

$$\frac{\partial \psi}{\partial y} = x^2 e^{3x} + e^{3x} y^2. \tag{2}$$

Integrate both sides of equation (2) partially with respect to y to get ψ.

$$\psi(x, y) = x^2 e^{3x} y + \frac{e^{3x} y^3}{3} + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 3x^2 e^{3x} y + 2x e^{3x} y + e^{3x} y^3 + f'(x)$$

Comparing this to equation (1), we see that

$$f'(x) = 0 \quad \rightarrow \quad f(x) = 0.$$

As a result, a potential function is

$$\psi(x, y) = x^2 e^{3x} y + \frac{e^{3x} y^3}{3}.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

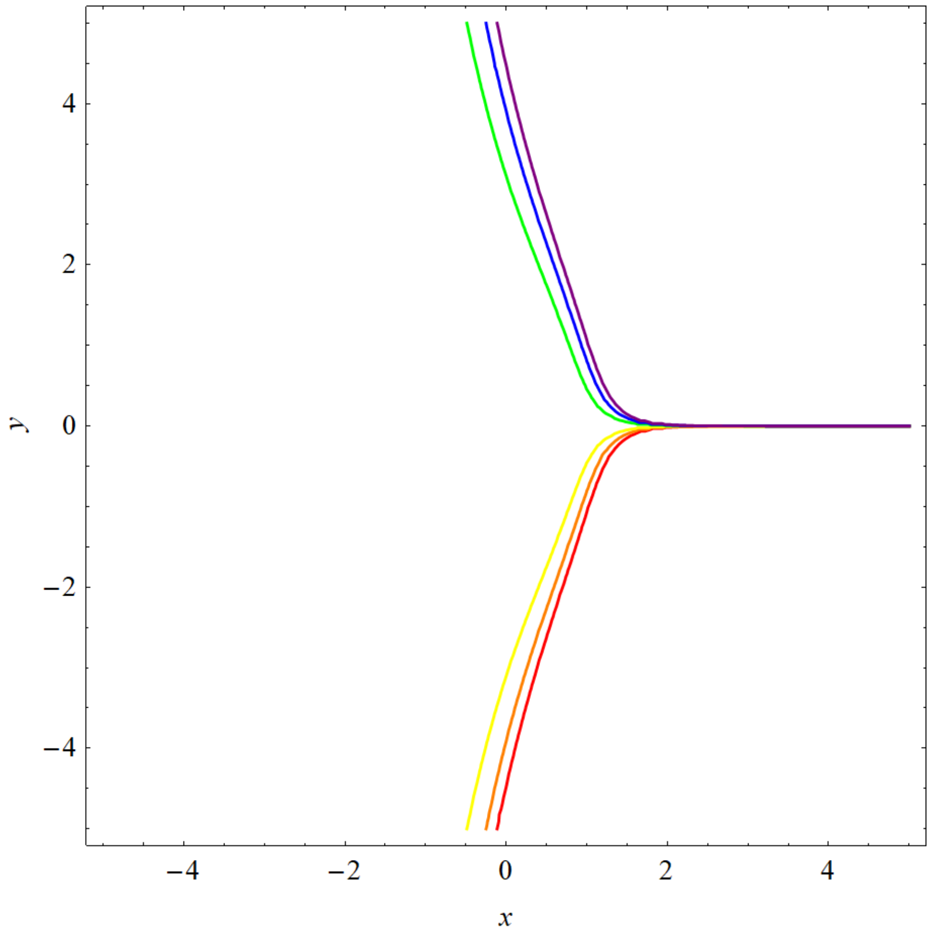
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$x^2 e^{3x} y + \frac{e^{3x} y^3}{3} = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -30$, $C = -20$, $C = -10$, $C = 10$, $C = 20$, and $C = 30$, respectively.