Problem 27

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

\[ 1 + (x/y - \sin y)y' = 0 \]

Solution

This ODE is not exact at the moment because

\[ \frac{\partial}{\partial y}(1) = 0 \neq \frac{\partial}{\partial x} \left( \frac{x}{y} - \sin y \right) = \frac{1}{y}. \]

To solve it, we seek an integrating factor \( \mu = \mu(x, y) \) such that when both sides are multiplied by it, the ODE becomes exact.

\[ \mu + \mu \left( \frac{x}{y} - \sin y \right) y' = 0 \]

Since the ODE is exact now,

\[ \frac{\partial}{\partial y}(\mu) = \frac{\partial}{\partial x} \left[ \mu \left( \frac{x}{y} - \sin y \right) \right]. \]

Expand both sides.

\[ \frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x} \left( \frac{x}{y} - \sin y \right) + \frac{\mu}{y} \]

Assume that \( \mu \) is only dependent on \( y \): \( \mu = \mu(y) \).

\[ \frac{d\mu}{dy} = \frac{\mu}{y} \]

Solve this ODE by separating variables.

\[ \frac{d\mu}{\mu} = \frac{dy}{y} \]

Integrate both sides.

\[ \ln \mu = \ln y + C \]

Exponentiate both sides.

\[ \mu = ye^C \]

Taking \( e^C \) to be 1, an integrating factor is

\[ \mu = y. \]

Multiply both sides of the original ODE by \( y \).

\[ y + (x - y \sin y)y' = 0 \]

Because it’s exact, there exists a potential function \( \psi = \psi(x, y) \) that satisfies

\[ \frac{\partial \psi}{\partial x} = y \quad (1) \]

\[ \frac{\partial \psi}{\partial y} = x - y \sin y. \quad (2) \]

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Integrate both sides of equation (1) partially with respect to $x$ to get $\psi$. 

$$\psi(x, y) = xy + f(y)$$

Here $f(y)$ is an arbitrary function of $y$. Differentiate both sides with respect to $y$.

$$\psi_y(x, y) = x + f'(y)$$

Comparing this to equation (2), we see that 

$$f'(y) = -y \sin y \rightarrow f(y) = y \cos y - \sin y.$$ 

As a result, a potential function is 

$$\psi(x, y) = xy + y \cos y - \sin y.$$ 

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.$$ 

(3)

Recall that the differential of $\psi(x, y)$ is defined as 

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$ 

Dividing both sides by $dx$, we obtain the fundamental relationship between the total derivative of $\psi$ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes 

$$\frac{d\psi}{dx} = 0.$$ 

Integrate both sides with respect to $x$.

$$\psi(x, y) = C_1$$ 

Therefore, 

$$xy + y \cos y - \sin y = C_1.$$
This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -10$, $C_1 = -5$, $C_1 = -1$, $C_1 = 1$, $C_1 = 5$, and $C_1 = 10$, respectively.