Problem 28

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$y + (2xy - e^{-2y})y' = 0$$

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(y) = 1 \neq \frac{\partial}{\partial x}(2xy - e^{-2y}) = 2y.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\mu y + \mu(2xy - e^{-2y})y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}(\mu y) = \frac{\partial}{\partial x}[\mu(2xy - e^{-2y})].$$

Expand both sides.

$$\frac{\partial \mu}{\partial y}y + \mu = \frac{\partial \mu}{\partial x}(2xy - e^{-2y}) + \mu(2y)$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$\frac{d\mu}{dy}y + \mu = \mu(2y)$$

$$\frac{d\mu}{dy} = \frac{2y - 1}{y}\mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \frac{2y - 1}{y}\,dy$$

Integrate both sides.

$$\ln \mu = 2y - \ln y + C$$

Exponentiate both sides.

$$\mu = e^{2y}\left(\frac{1}{y}\right)e^{C}$$

Taking e^C to be 1, an integrating factor is

$$\mu = \frac{e^{2y}}{y}.$$

Multiply both sides of the original ODE by e^{2y}/y.

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = e^{2y} \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 2xe^{2y} - \frac{1}{y}. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = xe^{2y} + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = 2xe^{2y} + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = -\frac{1}{y} \quad \rightarrow \quad f(y) = -\ln|y|.$$

As a result, a potential function is

$$\psi(x, y) = xe^{2y} - \ln|y|.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

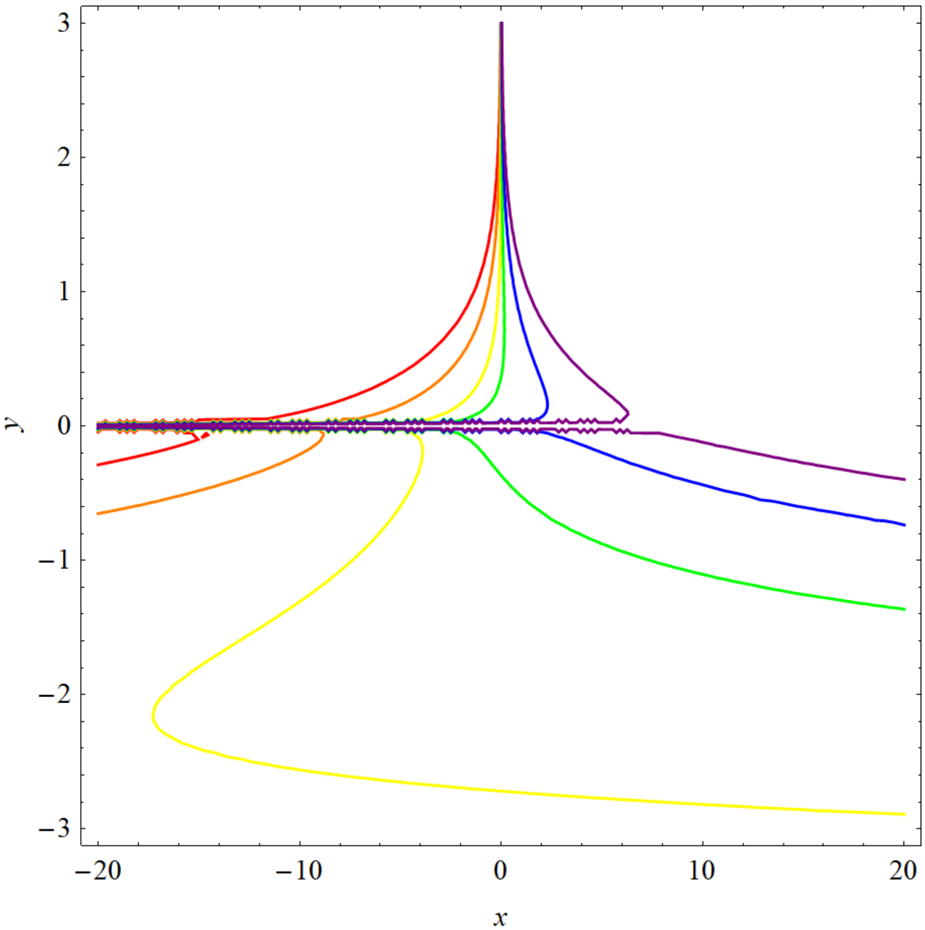
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

Therefore,

$$xe^{2y} - \ln|y| = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -10$, $C_1 = -5$, $C_1 = -1$, $C_1 = 1$, $C_1 = 5$, and $C_1 = 10$, respectively.

Notice also that $y = 0$ is a solution.