Problem 30

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$[4(x^3/y^2) + (3/y)] + [3(x/y^2) + 4y]y' = 0$$

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}\left(4\frac{x^3}{y^2} + \frac{3}{y}\right) = -8\frac{x^3}{y^3} - \frac{3}{y^2} \neq \frac{\partial}{\partial x}\left(3\frac{x}{y^2} + 4y\right) = \frac{3}{y^2}.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\mu + \mu\left(3\frac{x}{y^2} + 4y\right)y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}\left[\left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\mu\right] = \frac{\partial}{\partial x}\left[\mu\left(3\frac{x}{y^2} + 4y\right)\right].$$

Expand both sides.

$$\left(-8\frac{x^3}{y^3} - \frac{3}{y^2}\right)\mu + \left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}\left(3\frac{x}{y^2} + 4y\right) + \mu\left(\frac{3}{y^2}\right)$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$\left(-8\frac{x^3}{y^3} - \frac{3}{y^2}\right)\mu + \left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\frac{d\mu}{dy} = \mu\left(\frac{3}{y^2}\right)$$

$$\left(-8\frac{x^3}{y^3} - \frac{6}{y^2}\right)\mu + \left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\frac{d\mu}{dy} = 0$$

$$-\frac{2}{y}\left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\mu + \left(4\frac{x^3}{y^2} + \frac{3}{y}\right)\frac{d\mu}{dy} = 0$$

$$-\frac{2}{y}\mu + \frac{d\mu}{dy} = 0$$

$$\frac{d\mu}{dy} = \frac{2}{y}\mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \frac{2}{y}\,dy$$

Integrate both sides.

$$\ln\mu = 2\ln y + C$$

$$\ln\mu = \ln y^2 + C$$

Exponentiate both sides.

$$\mu = (y^2)e^C$$

Taking e^C to be 1, an integrating factor is

$$\mu = y^2.$$

Multiply both sides of the original ODE by y^2.

$$(4x^3 + 3y) + (3x + 4y^3)y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 4x^3 + 3y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 3x + 4y^3. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = x^4 + 3xy + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = 3x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 4y^3 \quad \to \quad f(y) = y^4.$$

As a result, a potential function is

$$\psi(x, y) = x^4 + 3xy + y^4.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

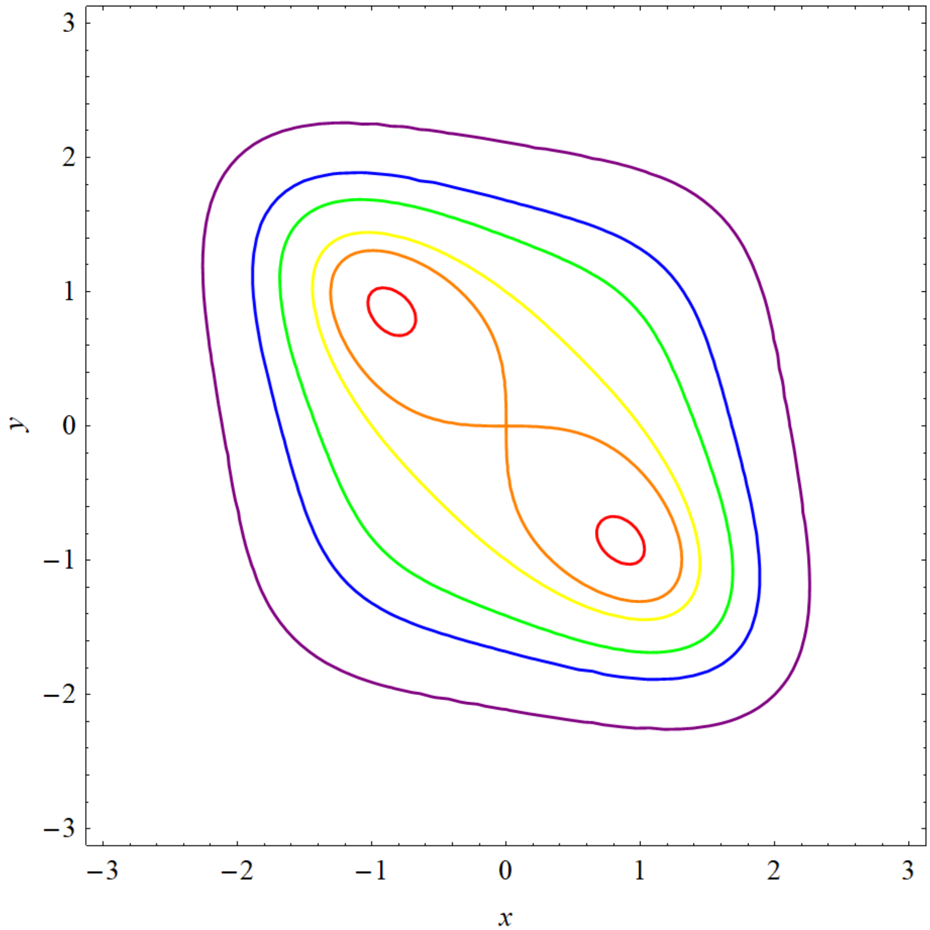
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

Therefore,

$$x^4 + 3xy + y^4 = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -1$, $C_1 = 0$, $C_1 = 1$, $C_1 = 4$, $C_1 = 8$, and $C_1 = 20$, respectively.