Problem 32

Solve the differential equation
\[(3xy + y^2) + (x^2 + xy)y' = 0\]
using the integrating factor \(\mu(x, y) = [xy(2x + y)]^{-1}\). Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

Solution

Multiply both sides of the ODE by \(\mu(x, y)\).
\[
\left[ \frac{3}{2x + y} + \frac{y}{x(2x + y)} \right] + \left[ \frac{x}{y(2x + y)} + \frac{1}{2x + y} \right] y' = 0
\]
This ODE is exact because
\[
\frac{\partial}{\partial y} \left[ \frac{3}{2x + y} + \frac{y}{x(2x + y)} \right] = \frac{\partial}{\partial x} \left[ \frac{x}{y(2x + y)} + \frac{1}{2x + y} \right] = -\frac{1}{(2x + y)^2}.
\]
That means there exists a potential function \(\psi = \psi(x, y)\) which satisfies
\[
\frac{\partial \psi}{\partial x} = \frac{3}{2x + y} + \frac{y}{x(2x + y)} \quad (1)
\]
\[
\frac{\partial \psi}{\partial y} = \frac{x}{y(2x + y)} + \frac{1}{2x + y} \quad (2)
\]
Integrate both sides of equation (1) partially with respect to \(x\) to get \(\psi\).
\[
\int x \frac{\partial \psi}{\partial x} \Bigg|_{x=s} \, ds = \int x \frac{3}{2s + y} \, ds + \int x \frac{y}{s(2s + y)} \, ds + f(y)
\]
Here \(f(y)\) is an arbitrary function of \(y\).
\[
\psi(x, y) = \int x \frac{3}{2s + y} \, ds + \int x \left( \frac{1}{s} + \frac{-2}{2s + y} \right) \, ds + f(y)
\]
\[
= \int x \frac{ds}{2s + y} + \int x \frac{ds}{s} + f(y)
\]
Use the substitution \(u = 2s + y\) and \(du = 2 \, ds\) in the first integral.
\[
\psi(x, y) = \int^{2x+y} \frac{du/2}{u} + \int^x \frac{ds}{s} + f(y)
\]
\[
= \frac{1}{2} \ln(2x + y) + \ln x + f(y)
\]
Differentiate both sides with respect to \(y\).
\[
\psi_y(x, y) = \frac{1}{2(2x + y)} + f'(y)
\]
Comparing this to equation (2), we see that
\[
\frac{1}{2(2x + y)} + f'(y) = \frac{x}{y(2x + y)} + \frac{1}{2x + y} \quad \rightarrow \quad f'(y) = \frac{x}{y(2x + y)} + \frac{1}{2(2x + y)} = \frac{2x + y}{2y(2x + y)} = \frac{1}{2y}
\]
which means
\[ f(y) = \frac{1}{2} \ln y. \]

As a result, a potential function is
\[ \psi(x, y) = \frac{1}{2} \ln(2x + y) + \ln x + \frac{1}{2} \ln y. \]

Notice that by substituting equations (1) and (2), the ODE can be written as
\[ \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \]  
(3)

Recall that the differential of \( \psi(x, y) \) is defined as
\[ d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy. \]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.
\[ \frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \]

With it, equation (3) becomes
\[ \frac{d\psi}{dx} = 0. \]

Integrate both sides with respect to \( x \).
\[ \psi(x, y) = C_1 \]
\[ \frac{1}{2} \ln(2x + y) + \ln x + \frac{1}{2} \ln y = C_1 \]
\[ \ln(2x + y) + 2 \ln x + \ln y = 2C_1 \]
\[ \ln(2x + y) + \ln x^2 + \ln y = 2C_1 \]
\[ \ln[(2x + y)(x^2)(y)] = 2C_1 \]
\[ x^2 y(2x + y) = e^{2C_1} \]
\[ 2x^3 y + x^2 y^2 = e^{2C_1} \]
\[ x^3 y + \frac{1}{2} x^2 y^2 = \frac{1}{2} e^{2C_1} \]

Therefore, using a new constant \( c \) for the right side,
\[ x^3 y + \frac{1}{2} x^2 y^2 = c, \]
which is the same answer obtained in Example 4 of the textbook.