Problem 6

In each of Problems 3 through 6, let \( \phi_0(t) = 0 \) and define \( \{ \phi_n(t) \} \) by the method of successive approximations

(a) Determine \( \phi_n(t) \) for an arbitrary value of \( n \).

(b) Plot \( \phi_n(t) \) for \( n = 1, \ldots, 4 \). Observe whether the iterates appear to be converging.

(c) Express \( \lim_{n \to \infty} \phi_n(t) = \phi(t) \) in terms of elementary functions; that is, solve the given initial value problem.

(d) Plot \( |\phi(t) - \phi_n(t)| \) for \( n = 1, \ldots, 4 \). For each of \( \phi_1(t), \ldots, \phi_4(t) \), estimate the interval in which it is a reasonably good approximation to the actual solution.

\[
y' = y + 1 - t, \quad y(0) = 0
\]

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to \( t \).

\[
\frac{dy}{dt} = y + 1 - t
\]

\[
\int_0^t \left. \frac{dy}{dt} \right|_{s=t} ds = \int_0^t \left[ y(s) + 1 - s \right] ds
\]

\[
y(t) - y(0) = \int_0^t \left[ y(s) + 1 - s \right] ds
\]

\[
y(t) = \int_0^t \left[ y(s) + 1 - s \right] ds
\]

Use the method of successive approximations to solve for \( y(t) \). Consider the iteration scheme,

\[
y_{n+1}(t) = \int_0^t \left[ y_n(s) + 1 - s \right] ds, \quad n \geq 0,
\]

taking \( y_0(t) = 0 \) for the zeroth approximation. As a result,

\[
y_1(t) = \int_0^t \left[ y_0(s) + 1 - s \right] ds = \int_0^t \left( 1 - s \right) ds = t - \frac{t^2}{2}
\]

\[
y_2(t) = \int_0^t \left[ y_1(s) + 1 - s \right] ds = \int_0^t \left( s - \frac{s^2}{2} + 1 - s \right) ds = t - \frac{t^3}{6}
\]

\[
y_3(t) = \int_0^t \left[ y_2(s) + 1 - s \right] ds = \int_0^t \left( s - \frac{s^3}{6} + 1 - s \right) ds = t - \frac{t^4}{24}
\]

\[
y_4(t) = \int_0^t \left[ y_3(s) + 1 - s \right] ds = \int_0^t \left( s - \frac{s^4}{24} + 1 - s \right) ds = t - \frac{t^5}{120}
\]

\[
y_5(t) = \int_0^t \left[ y_4(s) + 1 - s \right] ds = \int_0^t \left( s - \frac{s^5}{120} + 1 - s \right) ds = t - \frac{t^6}{720}
\]

\[
y_6(t) = \int_0^t \left[ y_5(s) + 1 - s \right] ds = \int_0^t \left( s - \frac{s^6}{720} + 1 - s \right) ds = t - \frac{t^7}{5040}
\]

\[
\vdots
\]
A formula for $y_n(t)$ can be deduced.

$$y_n(t) = t - \frac{t^{n+1}}{(n+1)!}$$

Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$y(t) = \lim_{n \to \infty} y_n(t)$$

$$= t - \lim_{n \to \infty} \frac{t^{n+1}}{(n+1)!}$$

$$= t$$

Not only is this the solution to the integral equation, but it also satisfies the initial value problem.

Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as $n$ increases.
$y_1(t), y_2(t), y_3(t), y_4(t), y_5(t), \text{ and } y_6(t) \text{ are good approximations to } y(t) \text{ only up to about } t = 1, t = 1.5, t = 2, t = 2.5, t = 3, \text{ and } t = 3.5, \text{ respectively.}$