Problem 8

In each of Problems 7 and 8, let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

(a) Determine $\phi_n(t)$ for an arbitrary value of n.

(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

(c) Show that the sequence $\{\phi_n(t)\}$ converges.

$$y' = t^2 y - t, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned} \frac{dy}{dt} &= t^2 y - t \\ \int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [s^2 y(s) - s]\, ds \\ y(t) - y(0) &= \int_0^t [s^2 y(s) - s]\, ds \\ y(t) &= \int_0^t [s^2 y(s) - s]\, ds \end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t [s^2 y_n(s) - s]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned} y_1(t) &= \int_0^t [s^2 y_0(s) - s]\, ds = \int_0^t (-s)\, ds = -\frac{t^2}{2} \\ y_2(t) &= \int_0^t [s^2 y_1(s) - s]\, ds = \int_0^t \left[s^2\left(-\frac{s^2}{2}\right) - s\right] ds = -\frac{t^2}{2} - \frac{t^5}{10} \\ y_3(t) &= \int_0^t [s^2 y_2(s) - s]\, ds = \int_0^t \left[s^2\left(-\frac{s^2}{2} - \frac{s^5}{10}\right) - s\right] ds = -\frac{t^2}{2} - \frac{t^5}{10} - \frac{t^8}{80} \\ y_4(t) &= \int_0^t [s^2 y_3(s) - s]\, ds = \int_0^t \left[s^2\left(-\frac{s^2}{2} - \frac{s^5}{10} - \frac{s^8}{80}\right) - s\right] ds = -\frac{t^2}{2} - \frac{t^5}{10} - \frac{t^8}{80} - \frac{t^{11}}{880} \\ y_5(t) &= \int_0^t [s^2 y_4(s) - s]\, ds = \int_0^t \left[s^2\left(-\frac{s^2}{2} - \frac{s^5}{10} - \frac{s^8}{80} - \frac{s^{11}}{880}\right) - s\right] ds = -\frac{t^2}{2} - \frac{t^5}{10} - \frac{t^8}{80} - \frac{t^{11}}{880} - \frac{t^{14}}{12\,320} \end{aligned}$$

$$y_6(t) = \int_0^t [s^2 y_5(s) - s]\, ds = \int_0^t \left[s^2\left(-\frac{s^2}{2} - \frac{s^5}{10} - \frac{s^8}{80} - \frac{s^{11}}{880} - \frac{s^{14}}{12\,320}\right) - s\right] ds = -\frac{t^2}{2} - \frac{t^5}{10} - \frac{t^8}{80} - \frac{t^{11}}{880} - \frac{t^{14}}{12\,320} - \frac{t^{17}}{209\,440}$$

$$\vdots$$

A formula for $y_n(t)$ can be deduced.

$$\begin{aligned} y_n(t) &= -\frac{t^2}{2} - \frac{t^5}{10} - \frac{t^8}{80} - \frac{t^{11}}{880} - \frac{t^{14}}{12\,320} - \frac{t^{17}}{209\,440} - \cdots \\ &= -\left(\frac{t^2}{2} + \frac{t^5}{2\cdot 5} + \frac{t^8}{2\cdot 5\cdot 8} + \frac{t^{11}}{2\cdot 5\cdot 8\cdot 11} + \frac{t^{14}}{2\cdot 5\cdot 8\cdot 11\cdot 14} + \frac{t^{17}}{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17} + \cdots\right) \\ &= -\sum_{i=1}^{n} \frac{t^{3i-1}}{2\cdot 5\cdot 8\cdots(3i-1)} \end{aligned}$$

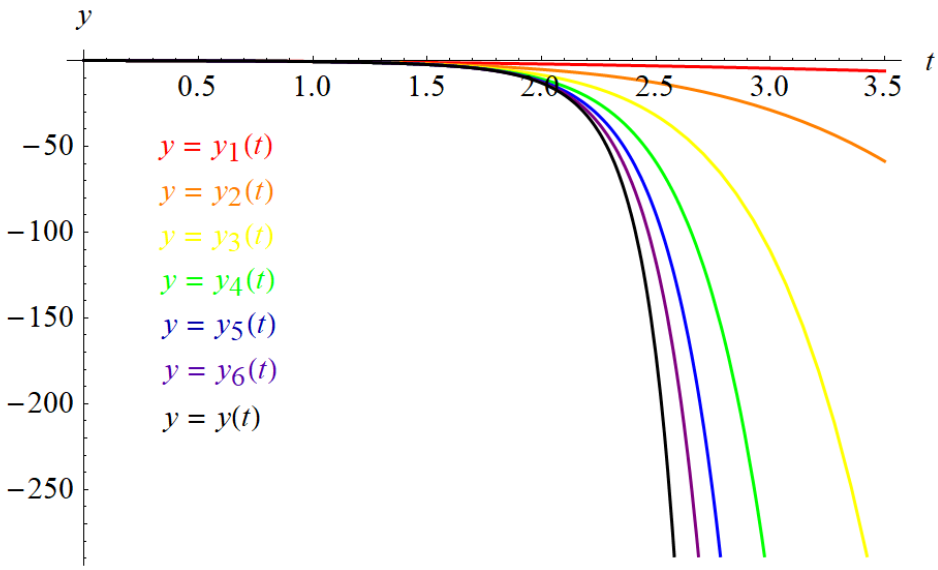
Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) \\ &= -\lim_{n\to\infty} \sum_{i=1}^{n} \frac{t^{3i-1}}{2\cdot 5\cdot 8\cdots(3i-1)} \\ &= -\sum_{i=1}^{\infty} \frac{t^{3i-1}}{2\cdot 5\cdot 8\cdots(3i-1)} \end{aligned}$$

We can show that $y(t)$ converges by using the ratio test.

$$\begin{aligned} \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\left|\frac{-\frac{t^{3(i+1)-1}}{2\cdot5\cdot8\cdots[3(i+1)-1]}}{-\frac{t^{3i-1}}{2\cdot5\cdot8\cdots(3i-1)}}\right| = \lim_{i\to\infty}\left|\frac{2\cdot 5\cdot 8\cdots(3i-1)}{2\cdot 5\cdot 8\cdots(3i+2)}\frac{t^{3i+2}}{t^{3i-1}}\right| &= \lim_{i\to\infty}\left|\frac{1}{(3i+2)}t^3\right| \\ &= t^3 \lim_{i\to\infty}\frac{1}{3i+2} \\ &= 0 \end{aligned}$$

Since the limit of this ratio is less than 1, the series solution for $y(t)$ converges (assuming finite t).



Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as n increases.