Problem 9

In each of Problems 9 and 10, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_3(t)$.

(b) Plot $\phi_1(t), \ldots, \phi_3(t)$ and observe whether the iterates appear to be converging.

$$y' = t^2 + y^2, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned}
\frac{dy}{dt} &= t^2 + y^2 \\
\int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [s^2 + y^2(s)]\, ds \\
y(t) - y(0) &= \int_0^t [s^2 + y^2(s)]\, ds \\
y(t) &= \int_0^t [s^2 + y^2(s)]\, ds
\end{aligned}$$

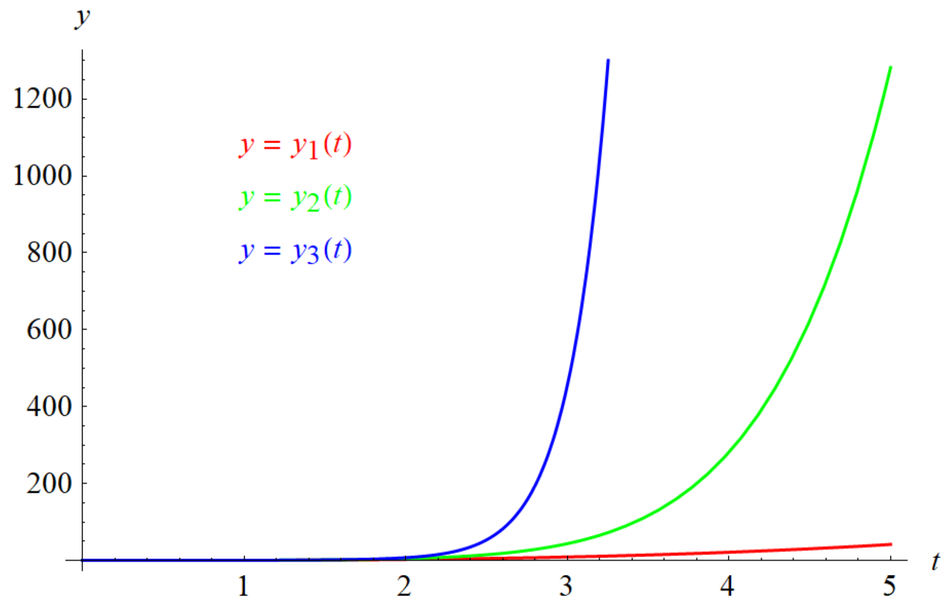
Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t [s^2 + y_n^2(s)]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned}
y_1(t) &= \int_0^t [s^2 + y_0^2(s)]\, ds = \int_0^t s^2\, ds = \frac{t^3}{3} \\
y_2(t) &= \int_0^t [s^2 + y_1^2(s)]\, ds = \int_0^t \left[s^2 + \left(\frac{s^3}{3}\right)^2\right] ds = \frac{t^3}{3} + \frac{t^7}{63} \\
y_3(t) &= \int_0^t [s^2 + y_2^2(s)]\, ds = \int_0^t \left[s^2 + \left(\frac{s^3}{3} + \frac{s^7}{63}\right)^2\right] ds = \frac{t^3}{3} + \frac{t^7}{63} + \frac{2t^{11}}{2079} + \frac{t^{15}}{59\,535} \\
&\vdots
\end{aligned}$$

Below is a plot of these approximations versus t.



They appear to converge.