Problem 10

In each of Problems 9 and 10, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_3(t)$.

(b) Plot $\phi_1(t), \ldots, \phi_3(t)$ and observe whether the iterates appear to be converging.

$$y' = 1 - y^3, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned} \frac{dy}{dt} &= 1 - y^3 \\ \int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [1 - y^3(s)]\, ds \\ y(t) - y(0) &= \int_0^t [1 - y^3(s)]\, ds \\ y(t) &= \int_0^t [1 - y^3(s)]\, ds \end{aligned}$$

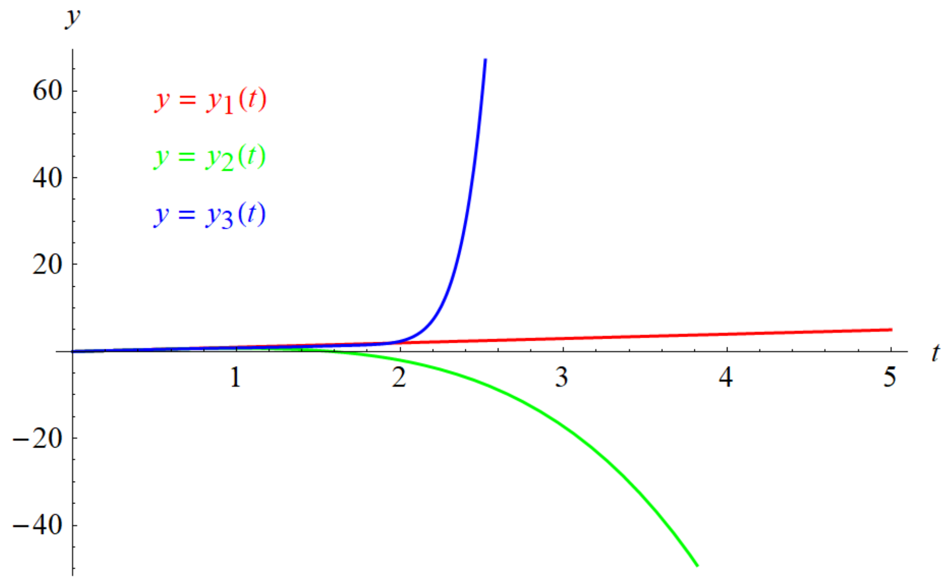
Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t [1 - y_n^3(s)]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned} y_1(t) &= \int_0^t [1 - y_0^3(s)]\, ds = \int_0^t ds = t \\ y_2(t) &= \int_0^t [1 - y_1^3(s)]\, ds = \int_0^t [1 - (s)^3]\, ds = t - \frac{t^4}{4} \\ y_3(t) &= \int_0^t [1 - y_2^3(s)]\, ds = \int_0^t \left[1 - \left(s - \frac{s^4}{4}\right)^3\right] ds = t - \frac{t^4}{4} + \frac{3t^7}{28} - \frac{3t^{10}}{160} + \frac{t^{13}}{832} \\ &\vdots \end{aligned}$$

Below is a plot of these approximations versus t.



They don't appear to converge.