Problem 11

In each of Problems 11 and 12, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_4(t)$, or (if necessary) Taylor approximations to these iterates. Keep terms up to order six.

(b) Plot the functions you found in part (a) and observe whether they appear to be converging.

$$y' = -\sin y + 1, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned}
\frac{dy}{dt} &= -\sin y + 1 \\
\int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [-\sin y(s) + 1]\, ds \\
y(t) - y(0) &= \int_0^t [-\sin y(s) + 1]\, ds \\
y(t) &= \int_0^t [-\sin y(s) + 1]\, ds
\end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

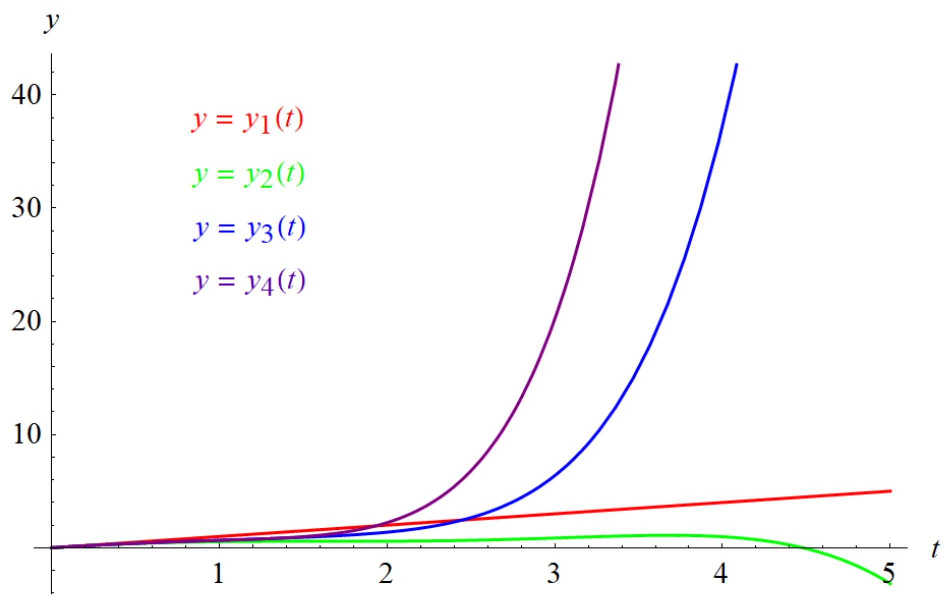
$$y_{n+1}(t) = \int_0^t [-\sin y_n(s) + 1]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned}
y_1(t) &= \int_0^t [-\sin y_0(s) + 1]\, ds = \int_0^t ds = t \\
y_2(t) &= \int_0^t [-\sin y_1(s) + 1]\, ds = \int_0^t (-\sin s + 1)\, ds = t + \cos t - 1 \\
&= t + \left(1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \cdots\right) - 1 = t - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \cdots \\
y_3(t) &= \int_0^t [-\sin y_2(s) + 1]\, ds = \int_0^t \left[-y_2(s) + \frac{y_2^3(s)}{6} - \frac{y_2^5(s)}{120} + \cdots + 1\right] ds \\
&= \int_0^t \left[-\left(s - \frac{s^2}{2} + \frac{s^4}{24} - \cdots\right) + \frac{1}{6}\left(s^3 - \frac{3}{2}s^4 + \frac{3}{4}s^5 - \cdots\right) - \frac{1}{120}\left(s^5 - \cdots\right) + \cdots + 1\right] ds \\
&= t - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} - \frac{7t^5}{120} + \frac{7t^6}{360} + \cdots
\end{aligned}$$

$$\begin{aligned} y_4(t) &= \int_0^t [-\sin y_3(s) + 1]\, ds = \int_0^t \left[-y_3(s) + \frac{y_3^3(s)}{6} - \frac{y_3^5(s)}{120} + \cdots + 1 \right] ds \\ &= \int_0^t \left[-\left(s - \frac{s^2}{2} + \frac{s^3}{6} + \frac{s^4}{24} - \frac{7s^5}{120} + \cdots \right) + \frac{1}{6}\left(s^3 - \frac{3}{2}s^4 + \frac{5}{4}s^5 - \cdots \right) - \frac{1}{120}\left(s^5 - \cdots\right) + \cdots + 1 \right] ds \\ &= t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{7t^5}{120} + \frac{31t^6}{720} + \cdots \end{aligned}$$

Below is a plot of these approximations (up to order 6) versus t.



They appear to converge.