Problem 3

In each of Problems 1 through 6, solve the given difference equation in terms of the initial value y_0. Describe the behavior of the solution as $n \to \infty$.

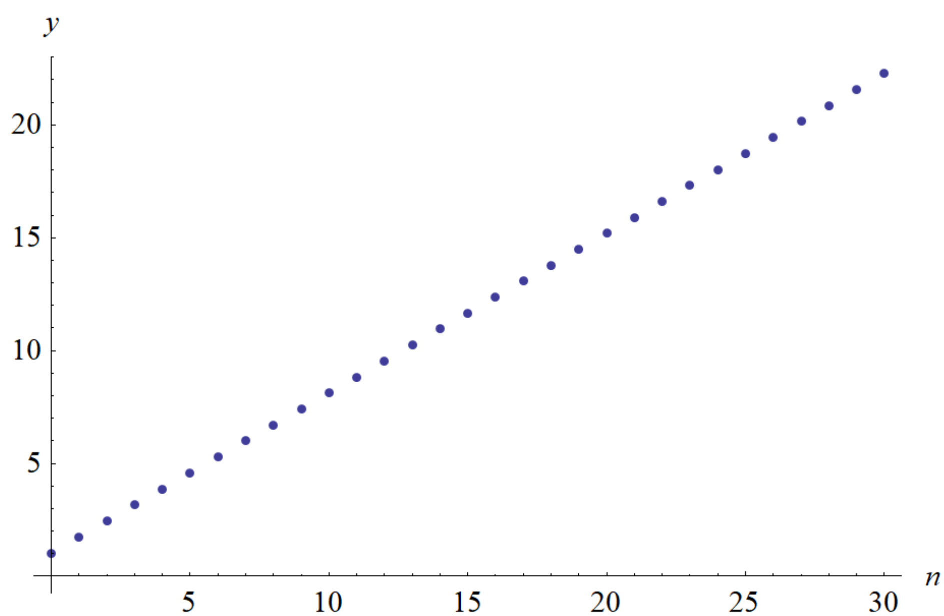
$$y_{n+1} = \sqrt{\frac{n+3}{n+1}} y_n$$

Solution

This is a first-order linear difference equation, so it can be solved by iteration.

$$\begin{aligned}
n = 0: \quad y_1 &= \sqrt{\frac{3}{1}} y_0 \\
n = 1: \quad y_2 &= \sqrt{\frac{4}{2}} y_1 = \sqrt{\frac{4}{2}} \sqrt{\frac{3}{1}} y_0 \\
n = 2: \quad y_3 &= \sqrt{\frac{5}{3}} y_2 = \sqrt{\frac{5}{3}} \sqrt{\frac{4}{2}} \sqrt{\frac{3}{1}} y_0 \\
&\vdots \\
y_n &= \frac{1}{\sqrt{2}} \sqrt{\frac{(n+2)!}{n!}} y_0 = \sqrt{\frac{(n+2)(n+1)}{2}} y_0
\end{aligned}$$

Below is a plot of y_n versus n for $y_0 = 1$.



Now take the limit of y_n as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \sqrt{\frac{(n+2)(n+1)}{2}} y_0 \\
&= \infty
\end{aligned}$$