Problem 5

In each of Problems 1 through 6, solve the given difference equation in terms of the initial value y_0. Describe the behavior of the solution as $n \to \infty$.

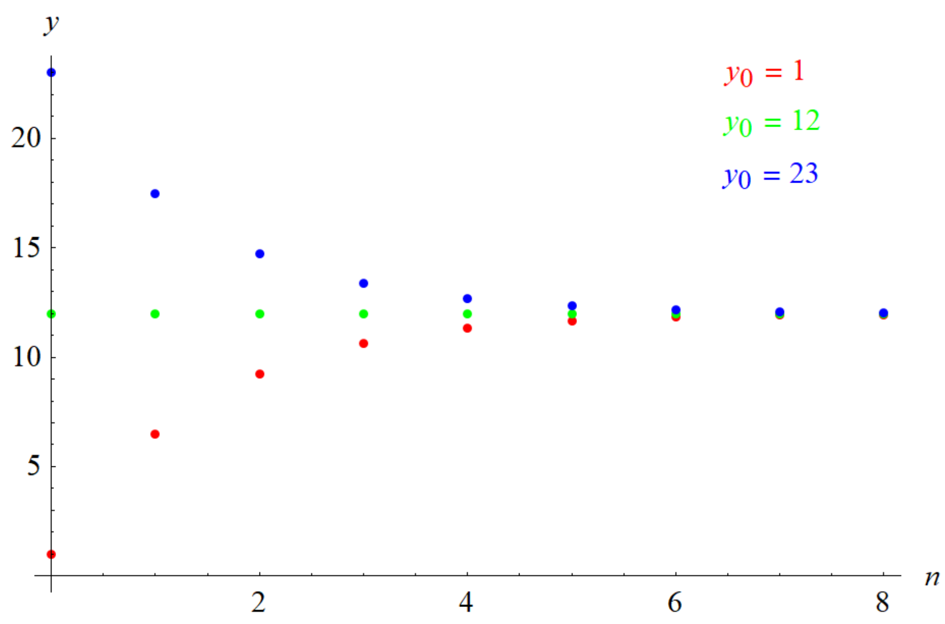
$$y_{n+1} = 0.5y_n + 6$$

Solution

This is a first-order linear difference equation, so it can be solved by iteration.

$$\begin{aligned}
n = 0: &\quad y_1 = 0.5y_0 + 6 \\
n = 1: &\quad y_2 = 0.5y_1 + 6 = 0.5(0.5y_0 + 6) + 6 = 0.5^2 y_0 + (0.5^1 + 1)6 \\
n = 2: &\quad y_3 = 0.5y_2 + 6 = 0.5[0.5(0.5y_0 + 6) + 6] + 6 = 0.5^3 y_0 + (0.5^2 + 0.5^1 + 1)6 \\
&\vdots \\
&\quad y_n = 0.5^n y_0 + (0.5^{n-1} + 0.5^{n-2} + \cdots + 1)6 \\
&\qquad = 0.5^n y_0 + \sum_{k=0}^{n-1} (0.5^k)6 \\
&\qquad = 0.5^n y_0 + \left(\frac{0.5^n - 1}{0.5 - 1}\right)6 \\
&\qquad = 0.5^n y_0 - (0.5^n)12 + 12 \\
&\qquad = 0.5^n (y_0 - 12) + 12
\end{aligned}$$

Below is a plot of y_n versus n for $y_0 = 1$, $y_0 = 12$, and $y_0 = 23$ in red, green, and blue, respectively.



Now take the limit of y_n as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} [0.5^n(y_0 - 12) + 12] \\
&= (y_0 - 12) \lim_{n\to\infty} 0.5^n + \lim_{n\to\infty} 12 \\
&= (y_0 - 12) \lim_{n\to\infty} e^{\ln 0.5^n} + 12 \\
&= (y_0 - 12) \lim_{n\to\infty} e^{n \ln 0.5} + 12 \\
&= (y_0 - 12) \lim_{n\to\infty} e^{-n \ln 2} + 12 \\
&= (y_0 - 12)e^{-\infty} + 12 \\
&= 12
\end{aligned}$$