Problem 6

In each of Problems 1 through 6, solve the given difference equation in terms of the initial value y_0. Describe the behavior of the solution as $n \to \infty$.

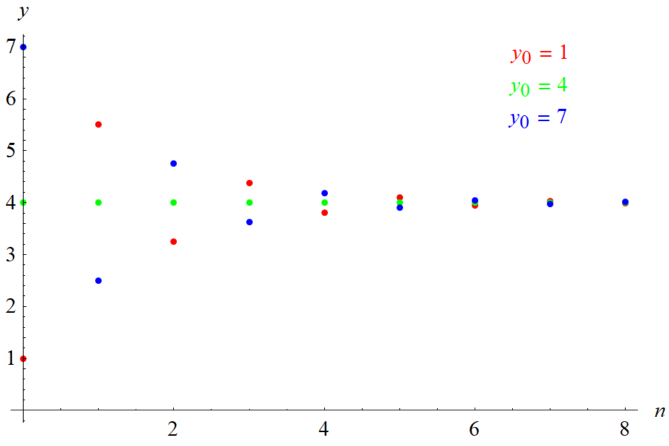
$$y_{n+1} = -0.5y_n + 6$$

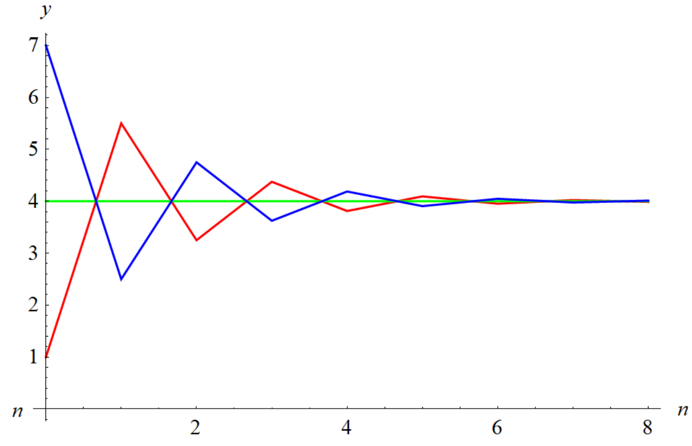
Solution

This is a first-order linear difference equation, so it can be solved by iteration.

$$\begin{aligned}
n = 0: &\quad y_1 = -0.5y_0 + 6 \\
n = 1: &\quad y_2 = -0.5y_1 + 6 = -0.5(-0.5y_0 + 6) + 6 = (-0.5)^2 y_0 + [(-0.5)^1 + 1]6 \\
n = 2: &\quad y_3 = -0.5y_2 + 6 = -0.5[-0.5(-0.5y_0 + 6) + 6] + 6 = (-0.5)^3 y_0 + [(-0.5)^2 + (-0.5)^1 + 1]6 \\
&\vdots \\
&\quad y_n = (-0.5)^n y_0 + [(-0.5)^{n-1} + (-0.5)^{n-2} + \cdots + 1]6 \\
&\quad\quad = (-0.5)^n y_0 + \sum_{k=0}^{n-1} (-0.5)^k 6 \\
&\quad\quad = (-0.5)^n y_0 + \left[\frac{(-0.5)^n - 1}{(-0.5) - 1}\right] 6 \\
&\quad\quad = (-0.5)^n y_0 - (-0.5^n)4 + 4 \\
&\quad\quad = (-0.5)^n (y_0 - 4) + 4
\end{aligned}$$

Below on the left is a plot of y_n versus n for $y_0 = 1$, $y_0 = 4$, and $y_0 = 7$ in red, green, and blue, respectively. On the right is the same graph but with the dots connected in order to better illustrate the solution's behavior.





Now take the limit of y_n as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} [(-0.5)^n (y_0 - 4) + 4] \\
&= (y_0 - 4) \lim_{n\to\infty} (-0.5)^n + \lim_{n\to\infty} 4 \\
&= (y_0 - 4) \left[\lim_{n\to\infty} (-1)^n\right] \left(\lim_{n\to\infty} 0.5^n\right) + 4 \\
&= (y_0 - 4) \left[\lim_{n\to\infty} (-1)^n\right] \left(\lim_{n\to\infty} e^{\ln 0.5^n}\right) + 4 \\
&= (y_0 - 4) \left[\lim_{n\to\infty} (-1)^n\right] \left(\lim_{n\to\infty} e^{n \ln 0.5}\right) + 4 \\
&= (y_0 - 4) \left[\lim_{n\to\infty} (-1)^n\right] \left(\lim_{n\to\infty} e^{-n \ln 2}\right) + 4 \\
&= (y_0 - 4) \left[\lim_{n\to\infty} (-1)^n\right] (e^{-\infty}) + 4 \\
&= 4
\end{aligned}$$