Problem 4

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = 3 - 6x + y - 2xy$$

Solution

Method Using an Integrating Factor I

Write the equation with dy/dx and y on the left side.

$$\frac{dy}{dx} + 2xy - y = 3 - 6x$$

$$\frac{dy}{dx} + (2x - 1)y = 3(1 - 2x)$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x (2s - 1)\, ds\right) = e^{x^2 - x}$$

Proceed with the multiplication.

$$e^{x^2-x}\frac{dy}{dx} + (2x - 1)e^{x^2-x}y = 3(1 - 2x)e^{x^2-x}$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(e^{x^2-x}y) = 3(1 - 2x)e^{x^2-x}$$

Integrate both sides with respect to x.

$$e^{x^2-x}y = \int^x 3(1 - 2s)e^{s^2-s}\, ds + C$$

Use the substitution,

$$\begin{aligned} u &= s^2 - s \\ du &= (2s - 1)\, ds = -(1 - 2s)\, ds. \end{aligned}$$

As a result,

$$\begin{aligned} e^{x^2-x}y &= \int^{x^2-x} 3e^u\,(-du) + C \\ &= -3\int^{x^2-x} e^u\, du + C \\ &= -3e^{x^2-x} + C. \end{aligned}$$

Therefore,

$$y(x) = -3 + Ce^{-x^2+x}.$$

Method Using an Integrating Factor II

$$\frac{dy}{dx} = 3 - 6x + y - 2xy$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(2xy - y + 6x - 3) + \frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(2xy - y + 6x - 3) = 2x - 1 \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(2xy - y + 6x - 3)\mu + \mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(2xy - y + 6x - 3)\mu] = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$(2x - 1)\mu + (2xy - y + 6x - 3)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$(2x - 1)\mu = \frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = (2x - 1)\,dx$$

Integrate both sides.

$$\ln \mu = x^2 - x + C_1$$

Exponentiate both sides.

$$\mu = e^{x^2 - x}e^{C_1}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^{x^2 - x}.$$

Multiply both sides of equation (1) by $e^{x^2 - x}$.

$$(2xe^{x^2 - x}y - ye^{x^2 - x} + 6xe^{x^2 - x} - 3e^{x^2 - x}) + e^{x^2 - x}\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x,y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2xe^{x^2 - x}y - ye^{x^2 - x} + 6xe^{x^2 - x} - 3e^{x^2 - x} \tag{3}$$

$$\frac{\partial \psi}{\partial y} = e^{x^2 - x}. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x,y) = e^{x^2-x}y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x,y) = (2x-1)e^{x^2-x}y + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = 6xe^{x^2-x} - 3e^{x^2-x} = 3(2x-1)e^{x^2-x} \quad \rightarrow \quad f(x) = 3e^{x^2-x}.$$

As a result, a potential function is

$$\psi(x,y) = e^{x^2-x}y + 3e^{x^2-x}.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

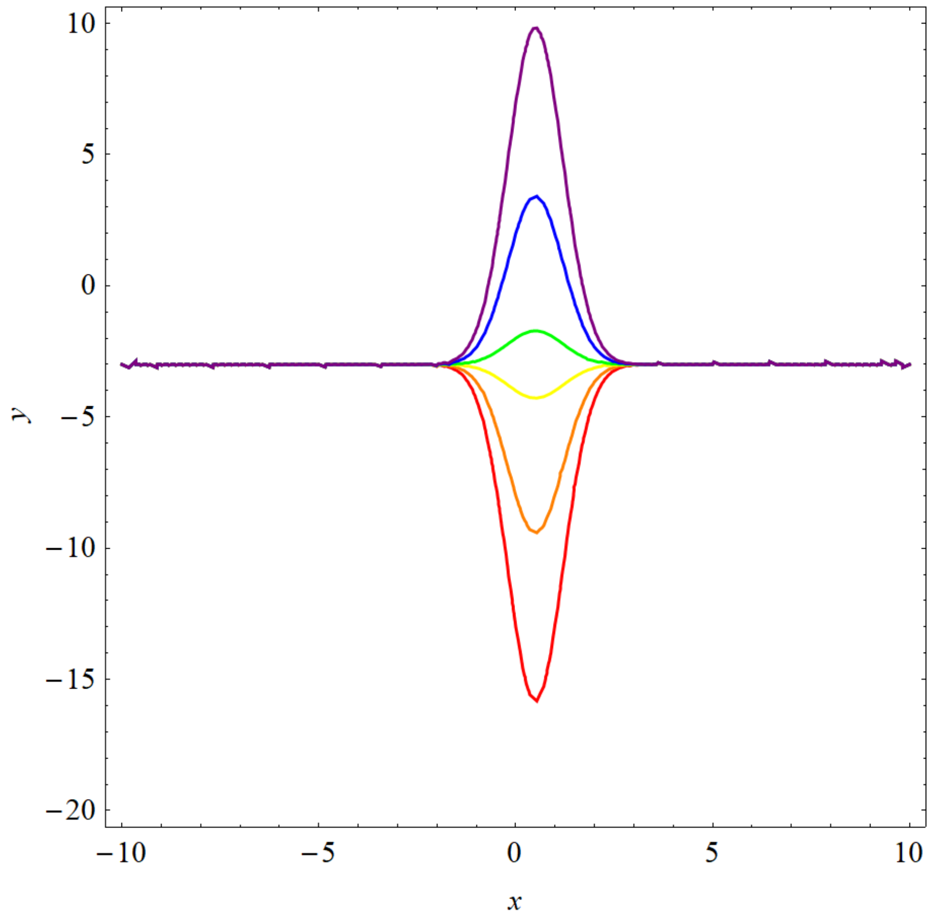
$$\psi(x,y) = C_2$$

Therefore,

$$e^{x^2-x}y + 3e^{x^2-x} = C_2,$$

or solving for y explicitly,

$$y(x) = -3 + C_2e^{-x^2+x}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.