

Problem 5

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$\begin{aligned}(x^2 + 2xy)\frac{dy}{dx} &= -(2xy + y^2 + 1) \\ (2xy + y^2 + 1) + (x^2 + 2xy)\frac{dy}{dx} &= 0\end{aligned}\tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(2xy + y^2 + 1) = \frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y.$$

That means there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial\psi}{\partial x} = 2xy + y^2 + 1\tag{2}$$

$$\frac{\partial\psi}{\partial y} = x^2 + 2xy.\tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ .

$$\psi(x, y) = x^2y + xy^2 + x + f(y)$$

Here $f(y)$ is an arbitrary function of y . Differentiate both sides with respect to y .

$$\psi_y(x, y) = x^2 + 2xy + f'(y)$$

Comparing this to equation (3), we see that

$$f'(y) = 0 \quad \rightarrow \quad f(y) = 0.$$

As a result, a potential function is

$$\psi(x, y) = x^2y + xy^2 + x.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0.\tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy.$$

Dividing both sides by dx , we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} \frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x .

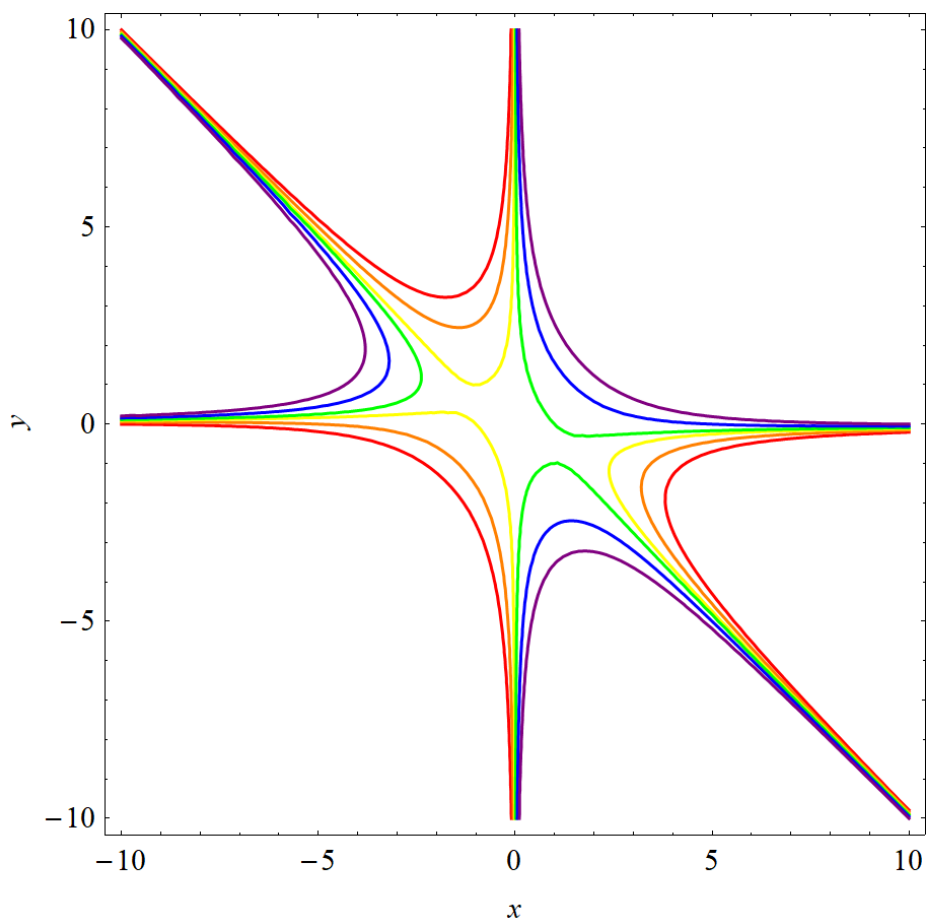
$$\psi(x, y) = C$$

Therefore,

$$x^2y + xy^2 + x = C,$$

or solving for y explicitly,

$$y(x) = \frac{-x^2 \pm \sqrt{x^4 - 4x(x - C)}}{2x}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.