Problem 6

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[ x \frac{dy}{dx} + xy = 1 - y, \quad y(1) = 0 \]

Solution

**Method Using an Integrating Factor I**

Bring \( y \) to the left and divide both sides by \( x \).

\[ \frac{dy}{dx} + \frac{x + 1}{x} y = \frac{1}{x} \]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int \frac{s + 1}{s} ds \right) = \exp \left( \int^x ds + \int^x \frac{ds}{s} \right) = e^{x + \ln x} = xe^x \]

Proceed with the multiplication.

\[ xe^x \frac{dy}{dx} + (x + 1)e^x y = e^x \]

The left side can be written as \( d/dx(Iy) \) by the chain rule.

\[ \frac{d}{dx}(xe^x y) = e^x \]

Integrate both sides with respect to \( x \).

\[ xe^x y = e^x + C \]

Apply the boundary condition \( y(1) = 0 \) now to determine \( C \).

\[ (1)e^1(0) = e^1 + C \quad \rightarrow \quad C = -e \]

As a result, the previous equation becomes

\[ xe^x y = e^x - e \]

\[ xy = 1 - e^{1-x}. \]

Therefore,

\[ y(x) = \frac{1 - e^{1-x}}{x}. \]
Method Using an Integrating Factor II

\[ x \frac{dy}{dx} + xy = 1 - y \]

Write the ODE as \( M(x, y) + N(x, y)y' = 0 \).

\[ (xy + y - 1) + x \frac{dy}{dx} = 0 \]  \hspace{1cm} (1)

This ODE is not exact at the moment because

\[ \frac{\partial}{\partial y} (xy + y - 1) = x + 1 \neq \frac{\partial}{\partial x} (x) = 1. \]

To solve it, we seek an integrating factor \( \mu = \mu(x, y) \) such that when both sides are multiplied by it, the ODE becomes exact.

\[ (xy + y - 1)\mu + x\mu \frac{dy}{dx} = 0 \]

Since the ODE is exact now,

\[ \frac{\partial}{\partial y} [(xy + y - 1)\mu] = \frac{\partial}{\partial x} (x\mu). \]

Expand both sides.

\[ (x + 1)\mu + (xy + y - 1) \frac{\partial \mu}{\partial y} = \mu + x \frac{\partial \mu}{\partial x} \]

Assume that \( \mu \) is only dependent on \( x \): \( \mu = \mu(x) \).

\[ (x + 1)\mu = \mu + x \frac{d\mu}{dx} \]

\[ x\mu = x \frac{d\mu}{dx} \]

\[ \frac{d\mu}{dx} = \mu \]

Solve this ODE by separating variables.

\[ \frac{d\mu}{\mu} = dx \]

Integrate both sides.

\[ \ln \mu = x + C_1 \]

Exponentiate both sides.

\[ \mu = e^{x+C_1} = e^x e^{C_1} \]

Taking \( e^{C_1} \) to be 1, an integrating factor is

\[ \mu = e^x. \]

Multiply both sides of equation (1) by \( e^x \).

\[ (xe^x y + e^x y - e^x) + xe^x \frac{dy}{dx} = 0 \]  \hspace{1cm} (2)

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Because it’s exact now, there exists a potential function \( \psi = \psi(x, y) \) that satisfies

\[
\frac{\partial \psi}{\partial x} = x e^x y + e^y - e^x \tag{3}
\]
\[
\frac{\partial \psi}{\partial y} = x e^x. \tag{4}
\]

Integrate both sides of equation (4) partially with respect to \( y \) to get \( \psi \).

\[
\psi(x, y) = x e^x y + f(x)
\]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[
\psi_x(x, y) = (x + 1)e^x y + f'(x)
\]

Comparing this to equation (3), we see that

\[
f'(x) = -e^x \quad \rightarrow \quad f(x) = -e^x.
\]

As a result, a potential function is

\[
\psi(x, y) = x e^x y - e^x.
\]

Notice that by substituting equations (3) and (4), equation (2) can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{5}
\]

Recall that the differential of \( \psi(x, y) \) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}
\]

With it, equation (5) becomes

\[
\frac{d\psi}{dx} = 0.
\]

Integrate both sides with respect to \( x \).

\[
\psi(x, y) = C_2
\]

The general solution is then

\[
x e^x y - e^x = C_2.
\]

Apply the boundary condition \( y(1) = 0 \) now to determine \( C_2 \).

\[
(1)e^1(0) - e^1 = C_2 \quad \rightarrow \quad C_2 = -e
\]

As a result, the previous equation becomes

\[
x e^x y - e^x = -e.
\]

\[
x e^x y = e^x - e.
\]

Therefore,

\[
y(x) = \frac{1 - e^{1-x}}{x}.
\]
This figure illustrates the solution to the ODE in the $xy$-plane that passes through the point $(1, 0)$. The solution is given by:

$$y(x) = \frac{1 - e^{1-x}}{x}$$