Problem 6

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$x\frac{dy}{dx} + xy = 1 - y, \qquad y(1) = 0$$

Solution

Method Using an Integrating Factor I

Bring y to the left and divide both sides by x.

$$\frac{dy}{dx} + \frac{x+1}{x}y = \frac{1}{x}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{s+1}{s}\,ds\right) = \exp\left(\int^x ds + \int^x \frac{ds}{s}\right) = e^{x+\ln x} = xe^x$$

Proceed with the multiplication.

$$xe^x\frac{dy}{dx} + (x+1)e^x y = e^x$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(xe^x y) = e^x$$

Integrate both sides with respect to x.

$$xe^x y = e^x + C$$

Apply the boundary condition $y(1) = 0$ now to determine C.

$$(1)e^1(0) = e^1 + C \quad \rightarrow \quad C = -e$$

As a result, the previous equation becomes

$$xe^x y = e^x - e$$

$$xy = 1 - e^{1-x}.$$

Therefore,

$$y(x) = \frac{1 - e^{1-x}}{x}.$$

Method Using an Integrating Factor II

$$x\frac{dy}{dx} + xy = 1 - y$$

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(xy + y - 1) + x\frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(xy + y - 1) = x + 1 \neq \frac{\partial}{\partial x}(x) = 1.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(xy + y - 1)\mu + x\mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(xy + y - 1)\mu] = \frac{\partial}{\partial x}(x\mu).$$

Expand both sides.

$$(x + 1)\mu + (xy + y - 1)\frac{\partial \mu}{\partial y} = \mu + x\frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$(x + 1)\mu = \mu + x\frac{d\mu}{dx}$$

$$x\mu = x\frac{d\mu}{dx}$$

$$\frac{d\mu}{dx} = \mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = dx$$

Integrate both sides.

$$\ln \mu = x + C_1$$

Exponentiate both sides.

$$\begin{aligned}\mu &= e^{x + C_1} \\ &= e^x e^{C_1}\end{aligned}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^x.$$

Multiply both sides of equation (1) by e^x.

$$(xe^x y + e^x y - e^x) + xe^x\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = xe^x y + e^x y - e^x \tag{3}$$

$$\frac{\partial \psi}{\partial y} = xe^x. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = xe^x y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = (x + 1)e^x y + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -e^x \quad \rightarrow \quad f(x) = -e^x.$$

As a result, a potential function is

$$\psi(x, y) = xe^x y - e^x.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_2$$

The general solution is then

$$xe^x y - e^x = C_2.$$

Apply the boundary condition $y(1) = 0$ now to determine C_2.

$$(1)e^1(0) - e^1 = C_2 \quad \rightarrow \quad C_2 = -e$$

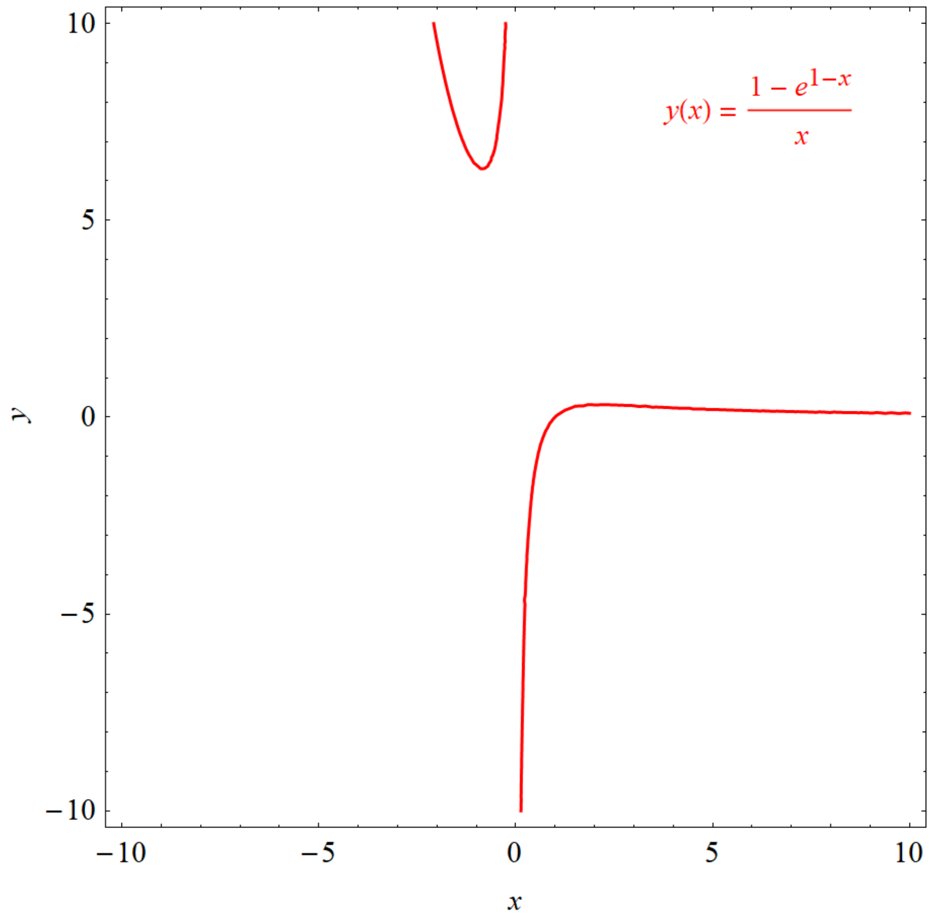
As a result, the previous equation becomes

$$xe^x y - e^x = -e.$$

$$xe^x y = e^x - e.$$

Therefore,

$$y(x) = \frac{1 - e^{1-x}}{x}.$$



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(1, 0)$.