Problem 7

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{4x^3 + 1}{y(2 + 3y)}$$

Solution

Method Using Separation of Variables

Because the ODE is of the form $y' = f(x)/g(y)$, it can be solved by separating variables.

$$y(2 + 3y)\,dy = (4x^3 + 1)\,dx$$

Integrate both sides.

$$\int (2y + 3y^2)\,dy = \int (4x^3 + 1)\,dx$$

$$y^2 + y^3 = x^4 + x + C$$

Therefore,

$$y^3 + y^2 - x^4 - x = C.$$

Method Using an Integrating Factor

$$\frac{dy}{dx} = \frac{4x^3+1}{y(2+3y)}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$y(2+3y)\frac{dy}{dx} = 4x^3+1$$

$$(-4x^3-1) + (2y+3y^2)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(-4x^3-1) = \frac{\partial}{\partial x}(2y+3y^2) = 0.$$

That means there exists a potential function $\psi = \psi(x,y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = -4x^3 - 1 \tag{2}$$

$$\frac{\partial \psi}{\partial y} = 2y + 3y^2. \tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ.

$$\psi(x,y) = -x^4 - x + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y) = f'(y)$$

Comparing this to equation (3), we see that

$$f'(y) = 2y + 3y^2 \quad \rightarrow \quad f(y) = y^2 + y^3.$$

As a result, a potential function is

$$\psi(x,y) = -x^4 - x + y^2 + y^3.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

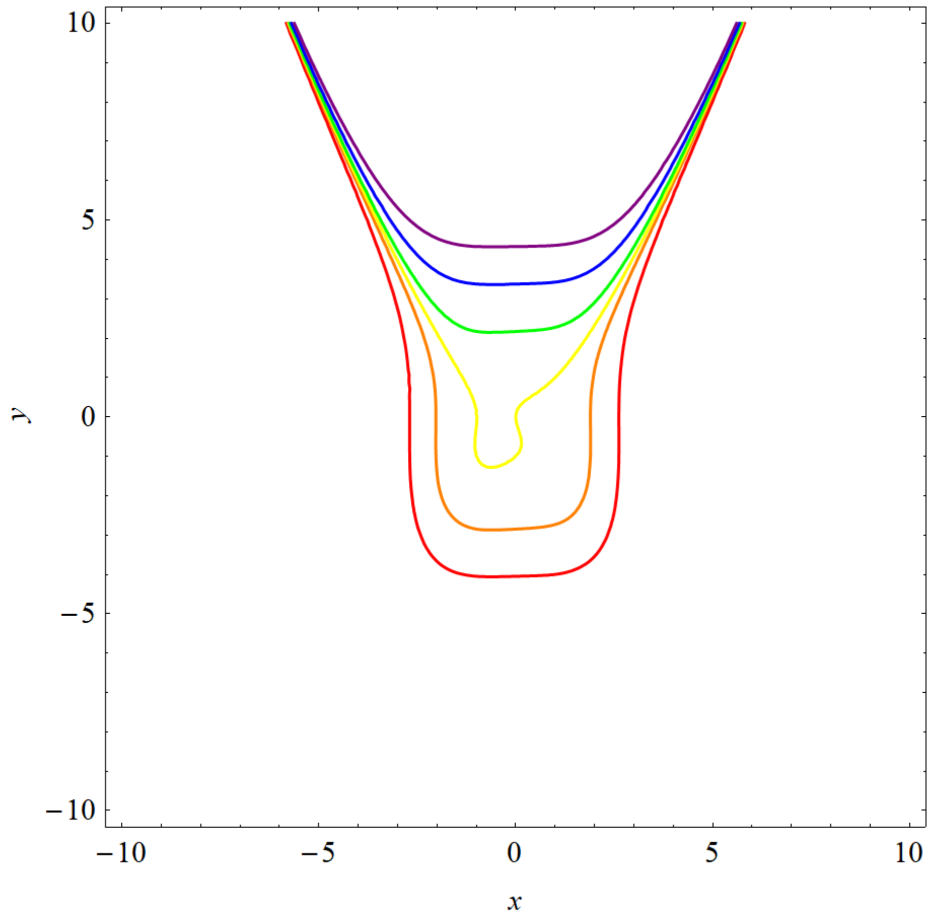
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x,y) = C_1$$

Therefore,

$$-x^4 - x + y^2 + y^3 = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -50$, $C = -15$, $C = 0$, $C = 15$, $C = 50$, and $C = 100$, respectively.