Problem 8

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$x\frac{dy}{dx} + 2y = \frac{\sin x}{x}, \qquad y(2) = 1$$

Solution

Method Using an Integrating Factor I

Divide both sides by x.

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x^2}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{2}{s}\,ds\right) = e^{2\ln x} = e^{\ln x^2} = x^2$$

Proceed with the multiplication.

$$x^2\frac{dy}{dx} + 2xy = \sin x$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(x^2 y) = \sin x$$

Integrate both sides with respect to x.

$$x^2 y = -\cos x + C$$

Apply the boundary condition $y(2) = 1$ now to determine C.

$$(2)^2(1) = -\cos 2 + C \quad \rightarrow \quad C = \cos 2 + 4$$

As a result, the previous equation becomes

$$x^2 y = -\cos x + \cos 2 + 4.$$

Therefore,

$$y(x) = \frac{-\cos x + \cos 2 + 4}{x^2}.$$

Method Using an Integrating Factor II

$$x\frac{dy}{dx} + 2y = \frac{\sin x}{x}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$\left(2y - \frac{\sin x}{x}\right) + x\frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}\left(2y - \frac{\sin x}{x}\right) = 2 \neq \frac{\partial}{\partial x}(x) = 1.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\left(2y - \frac{\sin x}{x}\right)\mu + x\mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}\left[\left(2y - \frac{\sin x}{x}\right)\mu\right] = \frac{\partial}{\partial x}(x\mu).$$

Expand both sides.

$$2\mu + \left(2y - \frac{\sin x}{x}\right)\frac{\partial \mu}{\partial y} = \mu + x\frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$2\mu = \mu + x\frac{d\mu}{dx}$$

$$\mu = x\frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \frac{dx}{x}$$

Integrate both sides.

$$\ln \mu = \ln x + C_1$$

Exponentiate both sides.

$$\mu = xe^{C_1}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = x.$$

Multiply both sides of equation (1) by x.

$$(2xy - \sin x) + x^2\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2xy - \sin x \tag{3}$$

$$\frac{\partial \psi}{\partial y} = x^2. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = x^2 y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 2xy + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -\sin x \quad \rightarrow \quad f(x) = \cos x.$$

As a result, a potential function is

$$\psi(x, y) = x^2 y + \cos x.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_2$$

The general solution is then

$$x^2 y + \cos x = C_2.$$

Apply the boundary condition $y(2) = 1$ now to determine C_2.

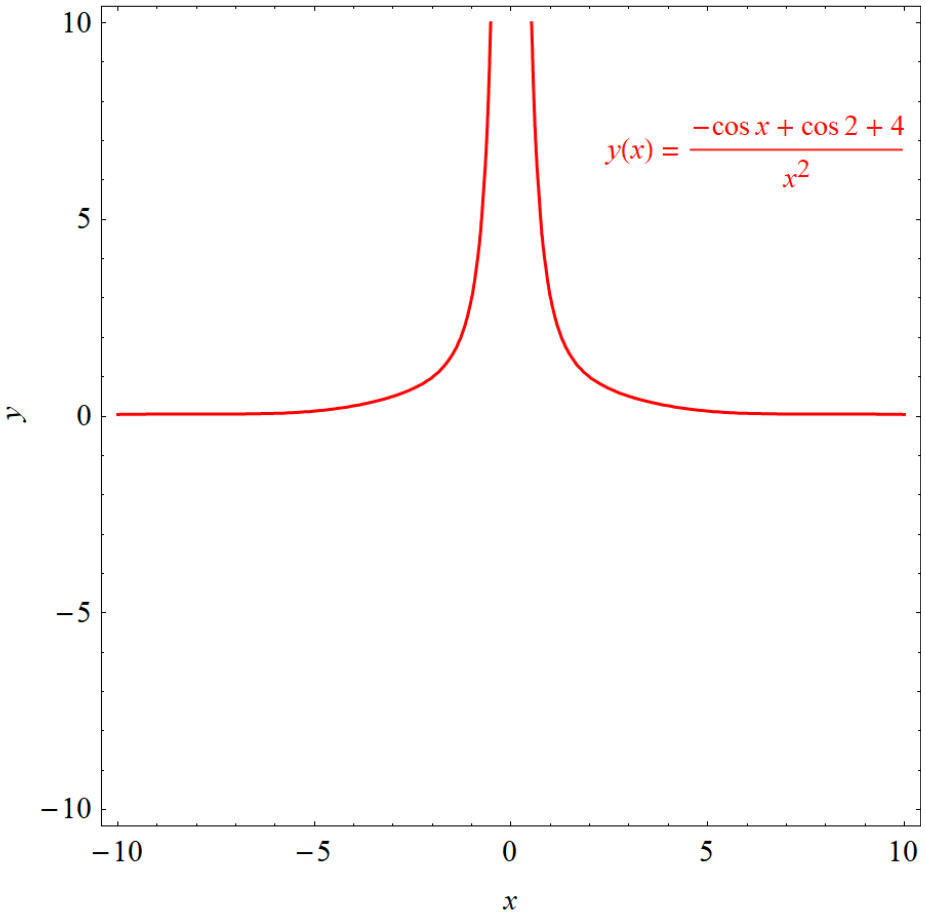
$$(2)^2(1) + \cos 2 = C_2 \quad \rightarrow \quad C_2 = 4 + \cos 2$$

As a result, the previous equation becomes

$$x^2 y + \cos x = 4 + \cos 2.$$

Therefore,

$$y(x) = \frac{-\cos x + \cos 2 + 4}{x^2}.$$



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(2, 1)$.