Problem 9

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[
\frac{dy}{dx} = -\frac{2xy + 1}{x^2 + 2y}
\]

Solution

Write the ODE as \( M(x,y) + N(x,y)y' = 0 \).

\[
(x^2 + 2y) \frac{dy}{dx} = -(2xy + 1)
\]

\[
(2xy + 1) + (x^2 + 2y) \frac{dy}{dx} = 0 \tag{1}
\]

This ODE is exact because

\[
\frac{\partial}{\partial y} (2xy + 1) = \frac{\partial}{\partial x} (x^2 + 2y) = 2x.
\]

That means there exists a potential function \( \psi = \psi(x,y) \) that satisfies

\[
\frac{\partial \psi}{\partial x} = 2xy + 1 \tag{2}
\]

\[
\frac{\partial \psi}{\partial y} = x^2 + 2y. \tag{3}
\]

Integrate both sides of equation (3) partially with respect to \( y \) to get \( \psi \).

\[
\psi(x,y) = x^2y + y^2 + f(x)
\]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[
\psi_x(x,y) = 2xy + f'(x)
\]

Comparing this to equation (2), we see that

\[
f'(x) = 1 \quad \rightarrow \quad f(x) = x.
\]

As a result, a potential function is

\[
\psi(x,y) = x^2y + y^2 + x.
\]

Notice that by substituting equations (2) and (3), equation (1) can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{4}
\]

Recall that the differential of \( \psi(x,y) \) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]
Dividing both sides by $dx$, we obtain the fundamental relationship between the total derivative of $\psi$ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to $x$.

$$\psi(x, y) = C$$

Therefore,

$$x^2 y + y^2 + x = C,$$

or solving for $y$ explicitly,

$$y(x) = -\frac{x^2 \pm \sqrt{x^4 - 4(x - C)}}{2}.$$

This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.