Problem 11

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[(x^2 + y) + (x + e^y) \frac{dy}{dx} = 0\]

Solution

This ODE is exact because

\[\frac{\partial}{\partial y} (x^2 + y) = \frac{\partial}{\partial x} (x + e^y) = 1.\]

That means there exists a potential function \(\psi = \psi(x,y)\) which satisfies

\[\frac{\partial \psi}{\partial x} = x^2 + y \quad (1)\]
\[\frac{\partial \psi}{\partial y} = x + e^y. \quad (2)\]

Integrate both sides of equation (2) partially with respect to \(y\) to get \(\psi\).

\[\psi(x,y) = xy + e^y + f(x)\]

Here \(f(x)\) is an arbitrary function of \(x\). Differentiate both sides with respect to \(x\).

\[\psi_x(x,y) = y + f'(x)\]

Comparing this to equation (1), we see that

\[f'(x) = x^2 \implies f(x) = \frac{x^3}{3}.\]

As a result, a potential function is

\[\psi(x,y) = xy + e^y + \frac{x^3}{3}.\]

Notice that by substituting equations (1) and (2), the original ODE can be written as

\[\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (3)\]

Recall that the differential of \(\psi(x,y)\) is defined as

\[d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.\]

Dividing both sides by \(dx\), we obtain the fundamental relationship between the total derivative of \(\psi\) and its partial derivatives.

\[\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}\]

With it, equation (3) becomes

\[\frac{d\psi}{dx} = 0.\]

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Integrate both sides with respect to $x$.

$$\psi(x, y) = C$$

Therefore,

$$xy + e^y + \frac{x^3}{3} = C.$$ 

This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.