Problem 15

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[(e^x + 1) \frac{dy}{dx} = y - ye^x\]

Solution

Method Using Separation of Variables

Factor the right side and then divide both sides by \(1 + e^x\).

\[\frac{dy}{dx} = \frac{y(1 - e^x)}{1 + e^x}\]

Because the ODE is of the form \(y' = f(x)g(y)\), it can be solved by separating variables.

\[\frac{dy}{y} = \frac{1 - e^x}{1 + e^x} \, dx\]

Integrate both sides.

\[\ln |y| = \int \frac{1 - e^s}{1 + e^s} \, ds + C\]

Make the following substitution.

\[u = e^s\]

\[du = e^s \, ds = u \, ds \rightarrow \frac{du}{u} = ds\]

As a result,

\[\ln |y| = \int_{e^x}^{e^u} \frac{1 - u}{1 + u} \, du + C\]

\[= \int_{e^x}^{e^u} \frac{du}{(1 + u)u} - \int_{e^x}^{e^u} \frac{du}{1 + u} + C\]

\[= \int_{e^x}^{e^u} \left( \frac{1}{u} - \frac{1}{1 + u} \right) \, du - \int_{e^x}^{e^u} \frac{du}{1 + u} + C\]

\[= \ln |u| \bigg|_{e^x}^{e^u} - \ln |1 + u| \bigg|_{e^x}^{e^u} - \ln |1 + u| + C\]

\[= \ln e^x - \ln(1 + e^x) - \ln(1 + e^x) + C\]

\[= x - 2 \ln(1 + e^x) + C\]

Bring the logarithms to the left side.

\[\ln |y| + \ln(1 + e^x)^2 = x + C\]

Combine them.

\[\ln |y|(1 + e^x)^2 = x + C\]

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Exponentiate both sides.

\[ |y|(1 + e^x)^2 = e^{x+C} \]

\[ |y| = \frac{e^C e^x}{(1 + e^x)^2} \]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[ y(x) = \frac{\pm e^C e^x}{(1 + e^x)^2} \]

Therefore, using a new constant \( A \) for \( \pm e^C \),

\[ y(x) = \frac{Ae^x}{(1 + e^x)^2} \]

\[ = \frac{A}{e^{-x}(1 + e^x)^2} \]

\[ = \frac{A}{(e^{-x/2}(1 + e^x/2))^2} \]

\[ = \frac{A}{4 \left( \frac{e^{x/2} + e^{-x/2}}{2} \right)^2} \]

\[ = \frac{A/4}{\cosh^2(x/2)} \]

\[ = \frac{B}{\cosh^2(x/2)} . \]
Method Using an Integrating Factor

\[(e^x + 1) \frac{dy}{dx} = y - ye^x\]

Bring all terms to the left side.

\[(e^x + 1) \frac{dy}{dx} + (e^x - 1)y = 0\]

Divide both sides by \(e^x + 1\).

\[\frac{dy}{dx} + \frac{e^x - 1}{e^x + 1}y = 0\]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \(I\).

\[I = \exp \left( \int \frac{e^s - 1}{e^s + 1} \, ds \right) = e^{2\ln(1+e^x) - x} = e^{\ln(1+e^x)^2}e^{-x} = (1 + e^x)^2e^{-x}\]

Proceed with the multiplication.

\[(e^x + 1)^2e^{-x} \frac{dy}{dx} + (e^x - 1)(e^x + 1)e^{-x}y = 0\]

\[(e^x + e^{-x} + 2) \frac{dy}{dx} + (e^x - e^{-x})y = 0\]

The left side can be written as \(d/dx[(e^x + e^{-x} + 2)y]\) by the chain rule.

\[\frac{d}{dx}[(e^x + e^{-x} + 2)y] = 0\]

Integrate both sides with respect to \(x\).

\[(e^x + e^{-x} + 2)y = C_1\]

Therefore,

\[y(x) = \frac{C_1}{e^x + e^{-x} + 2}\]

\[= \frac{C_1}{(e^{-x/2} + e^{x/2})^2}\]

\[= \frac{C_1}{4 \left( \frac{e^{x/2} + e^{-x/2}}{2} \right)^2}\]

\[= \frac{C_1}{4 \cosh^2(x/2)}\]

\[= \frac{C_1}{4 \cosh^2(x/2)} = \frac{C_2}{\cosh^2(x/2)}\].

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Method Using an Integrating Factor II

\[(e^x + 1) \frac{dy}{dx} = y - ye^x\]

Write the ODE as \(M(x, y) + N(x, y)y' = 0\).

\[(ye^x - y) + (e^x + 1) \frac{dy}{dx} = 0\] \hspace{1cm} (1)

This ODE is not exact at the moment because

\[\frac{\partial}{\partial y}(ye^x - y) = e^x - 1 \neq \frac{\partial}{\partial x}(e^x + 1) = e^x.\]

To solve it, we seek an integrating factor \(\mu(x, y)\) such that when both sides are multiplied by it, the ODE becomes exact.

\[(ye^x - y)\mu + (e^x + 1)\mu \frac{dy}{dx} = 0\]

Since the ODE is exact now,

\[\frac{\partial}{\partial y}[(ye^x - y)\mu] = \frac{\partial}{\partial x}[(e^x + 1)\mu].\]

Expand both sides.

\[(e^x - 1)\mu + (ye^x - y)\frac{\partial \mu}{\partial y} = e^x \mu + (e^x + 1)\frac{\partial \mu}{\partial x}\]

Assume that \(\mu\) is only dependent on \(x\): \(\mu = \mu(x)\).

\[(e^x - 1)\mu = e^x \mu + (e^x + 1)\frac{d\mu}{dx}\]

\[-\mu = (e^x + 1)\frac{d\mu}{dx}\]

Solve this ODE by separating variables.

\[\frac{d\mu}{\mu} = -\frac{dx}{e^x + 1}\]

Integrate both sides.

\[\ln \mu = -\int^x \frac{ds}{e^s + 1} + C_3\]

\[= -\int^x \frac{e^{-s}}{1 + e^{-s}} ds + C_3\]

Make the following substitution.

\[v = 1 + e^{-s}\]

\[dv = -e^{-s} ds\]

As a result,

\[\ln \mu = -\int^{1+e^{-s}} \frac{dv}{v} + C_3\]

\[= \ln(1 + e^{-x}) + C_3\]

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Exponentiate both sides.

\[ \mu = e^{\ln(1+e^{-x})+C_3} \]
\[ = (1 + e^{-x})e^{C_3} \]

Taking \(e^{C_3}\) to be 1, an integrating factor is

\[ \mu = 1 + e^{-x}. \]

Multiply both sides of equation (1) by \(1 + e^{-x}\).

\[(ye^x - y)(1 + e^{-x}) + (e^x + 1)(1 + e^{-x}) \frac{dy}{dx} = 0 \]
\[(e^x y - e^{-x} y) + (e^x + e^{-x} + 2) \frac{dy}{dx} = 0 \quad (2)\]

Because it’s exact now, there exists a potential function \(\psi = \psi(x, y)\) that satisfies

\[ \frac{\partial \psi}{\partial x} = e^x y - e^{-x} y \quad (3) \]
\[ \frac{\partial \psi}{\partial y} = e^x + e^{-x} + 2. \quad (4) \]

Integrate both sides of equation (4) partially with respect to \(y\) to get \(\psi\).

\[ \psi(x, y) = e^x y + e^{-x} y + 2y + h(x) \]

Here \(h(x)\) is an arbitrary function of \(x\). Differentiate both sides with respect to \(x\).

\[ \psi_x(x, y) = e^x y - e^{-x} y + h'(x) \]

Comparing this to equation (3), we see that

\[ h'(x) = 0 \quad \rightarrow \quad h(x) = 0. \]

Consequently, a potential function is

\[ \psi(x, y) = e^x y + e^{-x} y + 2y. \]

Notice that by substituting equations (3) and (4), equation (2) can be written as

\[ \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (5) \]

Recall that the differential of \(\psi(x, y)\) is defined as

\[ d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy. \]

Dividing both sides by \(dx\), we obtain the fundamental relationship between the total derivative of \(\psi\) and its partial derivatives.

\[ \frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \]
With it, equation (5) becomes
\[
\frac{d\psi}{dx} = 0.
\]
Integrate both sides with respect to \( x \).
\[
\psi(x, y) = C_4
\]
Therefore,
\[
e^x y + e^{-x} y + 2y = C_4,
\]
or solving for \( y \) explicitly,
\[
y(x) = \frac{C_4}{e^x + e^{-x} + 2} = \frac{C_4}{(e^{-x/2} + e^{x/2})^2} = \frac{C_4}{4 \left( \frac{e^{x/2} + e^{-x/2}}{2} \right)^2} = \frac{C_4/4}{\cosh^2(x/2)} = \frac{C_5}{\cosh^2(x/2)}.
\]

This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are \( C = -10 \), \( C = -5 \), \( C = -1 \), \( C = 1 \), \( C = 5 \), and \( C = 10 \), respectively.

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